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SECTION THREE

PRESIDENTIAL ADDRESS

Forbidden Transitions in the Spectra of Diatomic Molecules*

G. HERZBERG, F.R.S.C.

A. Introduction

THE study of forbidden transitions in atomic spectra has added a great deal to our understanding of atomic structure, and has contributed greatly to the solution of many astrophysical problems. Similariy the investigation of forbidden transitions in molecular spectra has proved to be of considerable interest, both for a better understanding of molecular structure and for their geophysical and astrophysical applications. Thus one of the most prominent features in the red part of the solar spectrum is now known to be due to a forbidden transition of the O_2 molecule in the earth's atmosphere.

My own interest in this subject dates back more than twenty years when I built my first long absorption tube for the study of forbidden transitions. In this address I should like to review some of the more recent results obtained by my collaborators and myself in the field of forbidden transitions in the spectra of diatomic molecules.

B. Elementary Facts about Molecular Spectra

In order to make my discussion intelligible to those not so familiar with the spectra of diatomic molecules, I should like to begin by summarizing very briefly a few elementary facts about the energy levels of diatomic molecules and the selection rules applying to them.

It is well known that the energy of a molecule can be resolved into three parts, the electronic energy E_e, the vibrational energy E_v, and the rotational energy E_r, i.e., we have

$$E = E_e + E_v + E_r = hc\,[T_e + G(v) + F_v(J)]$$

where the factor hc has been taken out in order to convert to term values in cm^{-1} units.

*Part of a symposium on Forbidden Transitions.

A molecule, just as an atom, has a series of *electronic states*, but unlike the atom, the electronic states of a diatomic molecule are classified according to the magnitude of the *component of the electronic orbital angular momentum in the direction of the inter-nuclear axis*. According to whether this component, designated by Λ, is 0, 1, 2, . . . (in units $h/2\pi$) the states are called Σ, Π, Δ, . . .states (corresponding to S, P, D, . . . of the atom). Two kinds of Σ states are distinguished, Σ^+ and Σ^-; the former corresponds to the case in which the electronic eigenfunction ψ_e upon reflection at a plane through the internuclear axis remains unchanged, and the latter corresponds to the case where it changes sign.

The *multiplicity* of the electronic states of a diatomic molecule is entirely similar to that of atoms, i.e., if the resultant spin of the electrons is S the multiplicity is $2S + 1$. Thus we have, e.g., a $^3\Delta$ state meaning a state with $\Lambda = 2$ and $S = 1$.

In the case of molecules consisting of two equal atoms (*homonuclear molecules*) like H_2, N_2, O_2, etc., an additional symmetry property must be distinguished. An electronic state of such a molecule is either *even* (indicated by the subscript g) when the electronic eigenfunction remains unchanged upon reflection at the centre, or it is *odd* (indicated by a subscript u) if the eigenfunction changes sign for such a reflection.

In each electronic state the molecule may have a number of *vibrational levels* whose position in a first approximation is given by the simple formula

$$G(v) = \omega_e(v + \tfrac{1}{2}) - \omega_e x_e(v + \tfrac{1}{2})^2$$

where $\omega_e c$ is the (classical) vibrational frequency, $\omega_e x_e \ll \omega_e$ is a small correction term due to the anharmonicity of the vibrations, and v is the vibrational quantum number. According to the formula, the levels are almost equidistant as shown in Fig. 1 (long horizontal lines).

In each vibrational level there is series of *rotational levels* given by the formula

$$F_v(J) = B_v J(J + 1)$$

where the rotational quantum number J assumes the value 0, 1, 2, etc., and represents the total angular momentum in units $h/2\pi$, and where B_v is the rotational constant in the vibrational level v. The rotational levels are not equidistant, but their separation increases with J since $F_v(J)$ is a quadratic function of J. This is also shown in Fig. 1 (short horizontal lines).

The rotational levels are alternatively "*positive*" and "*negative*" depending on whether the total eigenfunction of the molecule remains unchanged or changes sign upon reflection at the origin. In the simplest

case of a Σ^+ state the even rotational levels are "positive," the odd rotational levels are "negative." The reverse is the case in a Σ^- state. In Π, Δ, . . . states there are, for each J value, two levels, one "positive" and one "negative."

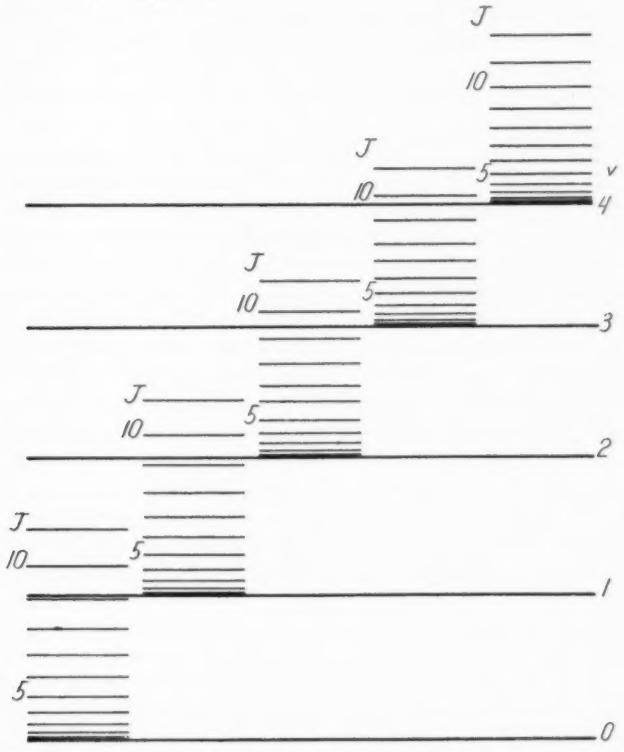

FIGURE 1.—Energy levels of a vibrating and rotating molecule in a given electronic state. The long horizontal lines represent the vibrational levels without rotation ($J = 0$), the short horizontal lines represent the rotational levels for each vibrational level.

In the case of homonuclear molecules we have to distinguish in addition *symmetric* and *antisymmetric* rotational levels corresponding to the behaviour of the eigenfunction with respect to an exchange of the two identical nuclei. For even electronic states (Σ_g, Π_g, . . .) the "positive" rotational levels are symmetric and the "negative" are antisymmetric; for odd electronic states (Σ_u, Π_u, . . .) the reverse is the case.

In the case of doublet, triplet, etc. electronic states one has two, three, etc. series of such rotational levels. The spacing of these doublets and triplets as a function of the rotational quantum number depends on the coupling conditions which we shall not discuss here. However, one special case should be mentioned: if the coupling of the electron spin with the rest of the molecule is very slight the rotational levels are essentially given by a formula as before except that J is replaced by K which is the quantum number of the angular momentum apart from spin.

For the transitions between the various energy levels of a diatomic molecule a number of *selection rules* hold. We may distinguish rigorous selection rules, which hold rigorously for dipole radiation of the free molecule, and approximate selection rules, whose validity depends on the coupling conditions in the molecule.

Rigorous selection rules are the following: the selection rule for the total angular momentum $\Delta J = 0, \pm 1$ (with the restriction that $J = 0$ does not combine with $J = 0$); furthermore, the *parity rule* (corresponding to the Laporte rule for atoms) which says that "positive" rotational levels combine only with "negative" rotational levels (or, for short, $+ \leftrightarrow -$). In the case of homonuclear molecules we have in addition the rule that symmetric rotational levels do not combine with antisymmetric ones (the ortho-para rule, for short, $s \leftarrow\!|\!\rightarrow a$). Finally, a selection rule may be added which is not independent of those already mentioned, namely, that the dipole moment must be different from zero in order that a pure rotation or a rotation-vibration spectrum can occur.

Examples of *approximate selection rules* are: the rule that states of different multiplicity do not combine with one another (that is, $\Delta S = 0$); the selection rule for the quantum number Λ of the electronic orbital angular momentum $\Delta \Lambda = 0, \pm 1$; and the selection rule for the rotational quantum number K, $\Delta K = 0, \pm 1$. Three other selection rules of this type which are important for the following considerations are the rules that Σ^+ states do not combine with Σ^- states, that even electronic states of molecules with nuclei of equal charge combine only with odd, and that $\Delta K = 0$ does not occur for Σ—Σ transitions.

C. Types and Observations of Forbidden Transitions

Two types of forbidden transitions may be distinguished.

1. *Forbidden transitions that violate approximate selection rules.* Examples in atoms are the intercombination lines such as the line λ2537 of mercury.

2. *Forbidden transitions that violate selection rules that are rigorous for dipole radiation in the free molecule.* Such violations may occur for two reasons:

(*a*) In the free molecule the transition may occur as *magnetic dipole, quadrupole,* or *higher pole radiation.* Well-known examples in the case of atomic spectra are the nebular, auroral, and coronal lines.

(*b*) The transitions may be *induced by the presence of external fields* or *by collisions with other molecules.* Examples in the case of atomic spectra are the helium lines with $\Delta L = 2$ which occur in strong electric fields.

Forbidden transitions in diatomic molecules of types 1 and 2(*a*) may be observed *in emission* under special conditions of excitation, particularly at extremely low pressure as it occurs for example in the upper atmosphere, but also in certain kinds of discharge tubes, either at low pressure, or when an inert gas is added. Forbidden transitions of type 2(*b*) require, of course, the presence of external fields or high pressure to be observed. *In absorption* forbidden transitions may be observed by using a very long absorbing path or high pressure or both.

In my laboratory we found the method first suggested by White [1] for the production of a very long absorbing path to be the most satisfactory [2]. In this method by means of three mirrors the light is made to traverse the absorption tube a large number of times. At the Yerkes Observatory I set up an absorption tube of 22 m. length and 25 cm. diameter with which it was possible by means of White's method to obtain up to 250 traversals, i.e., an absorbing path of 5500 m. At the National Research Laboratories in Ottawa we have set up a somewhat smaller tube of 5 m. length in which we have been able to obtain up to 200 traversals corresponding to a path length of 1000 m. In addition, in order to study absorption spectra at high pressures, that is, in order to study forbidden transitions of type 2(*b*) we have built a steel absorption cell of 2 m. length with similar mirrors by means of which we can obtain up to 60 traversals, that is, an absorbing path of 120 m. This tube will stand a pressure of 100 atm. and the whole tube may be immersed in liquid nitrogen in order to study the effect of temperature on these transitions.

D. Forbidden Rotation-Vibration Spectra

We shall first consider those forbidden transitions which do not involve a change of electronic state, that is, we shall consider forbidden rotation-vibration spectra of diatomic molecules. In a way the overtone bands of polar molecules, i.e., the transitions from the lowest vi-

brational level to higher vibrational levels with $v = 2, 3, \ldots$ may be considered as forbidden transitions although this is usually not done. They violate the selection rule for the harmonic oscillator, that is, $\Delta v = \pm 1$. Accordingly at least the higher ones of these transitions with $\Delta v = 3, 4$ can only be obtained with very long absorbing paths. Thus, for example, the 4—0 band of CO requires an absorbing path of 800 m. (see [3]).

Homonuclear molecules have no dipole moment and therefore they do not have an ordinary infrared rotation-vibration specrum. This is also excluded by the selection rules $s \leftarrow|\rightarrow a$ and $+ \leftrightarrow -$ since without change of electronic state the transitions allowed by the first

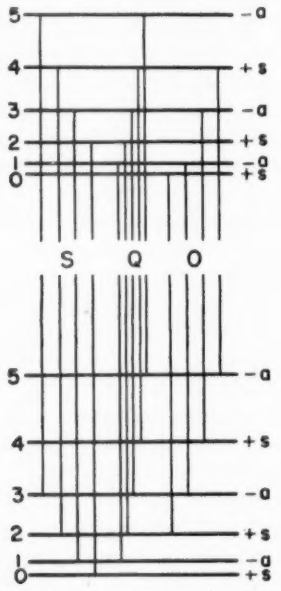

FIGURE 2.—Energy level diagram for the quadrupole rotation-vibration spectrum of a homonuclear molecule.

rule are just the ones rigorously forbidden by the second and conversely. However, a homonuclear molecule does have a *quadrupole moment* which will change during the vibration and may therefore cause the appearance of an extremely weak rotation-vibration spectrum. This is a forbidden transition of the type $2(a)$ made possible by quadrupole radiation. The selection rules for quadrupole radiation are

$$\Delta J = 0, \pm 1, \pm 2 \text{ and } + \leftarrow|\rightarrow -.$$

In the case of a transition between the rotational levels of two vi-

brational levels of a homonuclear molecule, as shown in Fig. 2, the transitions with $\Delta J = \pm 1$ are eliminated both by the rule $+ \leftarrow|\rightarrow -$ and the rule that $s \leftarrow|\rightarrow a$. Therefore, one expects three series of lines or branches, one with $\Delta J = 0$, a Q branch, one with $\Delta J = +2$, an S branch, and one with $\Delta J = -2$, an O branch. In the case of the H_2 molecule, on account of its very small moment of inertia (that is, of the wide spacing of the rotational levels), only very few lines in each of these branches are expected to occur at room temperature, and even fewer lines at lower temperatures. Fig. 3 shows the expected intensity distribution in the 2—0 band for a temperature of 150°K. This temperature was chosen fourteen years ago [4], when this figure

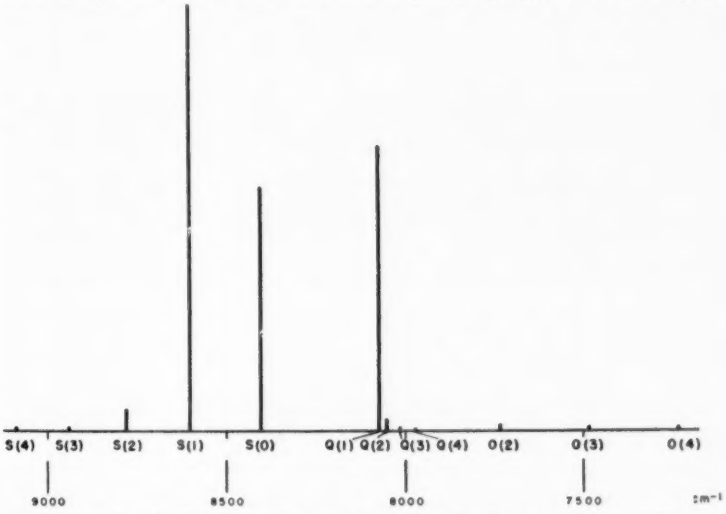

FIGURE 3.—Predicted structure and intensity distribution of the 2—0 quadrupole band of H_2 at 150°K. The height of the vertical lines indicates the predicted intensity.

was prepared, in order to be able to compare with the spectrum of the major planets in which large amounts of hydrogen were expected. James and Coolidge [5] have made more detailed calculations of the absolute intensities to be expected for these transitions and on the basis of their calculations one finds that the 1—0, 2—0, and 3—0 bands would require 2.5, 2.7, and 13.0 km. path at atmospheric pressure respectively.

With the absorption tube at Yerkes Observatory, which I mentioned previously, it appeared possible to observe the *quadrupole spectrum of hydrogen* since by increasing the pressure to 10 atm. an equivalent path of up to 55 km. atm. could be obtained. When this was done the

above prediction was indeed verified in that the 2—0 band was observed with a path length of 10 km. atm. and the 3—0 band with a path length of 40 km. atm. [6]. Fig. 4 shows several of the observed lines. Unfortunately, the spectral region in which these lines occur is overlapped by absorption due to water vapour and therefore in each case a comparison spectrum without hydrogen is shown. In this way the quadrupole lines of hydrogen can easily be identified. It should be particularly noted that in spite of a pressure of about 10 atm. the quadrupole lines are extremely sharp. Their width was as small as could be obtained in the second order of a 21 ft. grating.

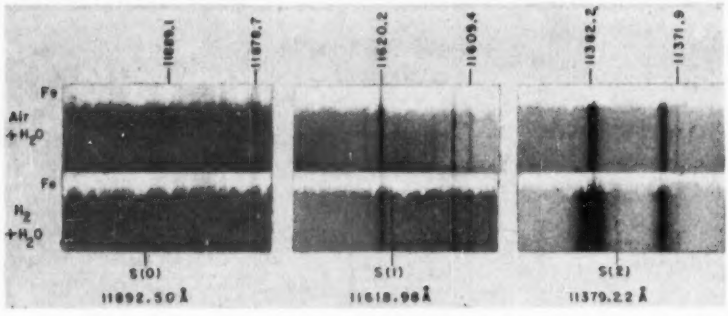

FIGURE 4.—Lines of the 2—0 quadrupole band of H_2. The top strips show for comparison the spectrum of moist air. In this way the quadrupole lines of H_2 in the bottom strips can be identified in spite of the overlapping H_2O spectrum.

A slight systematic shift was observed between the predicted positions of the quadrupole lines (predicted from the rotational and vibrational constants of hydrogen derived from its ultraviolet spectrum). This shift is due to the low accuracy obtainable in the far ultraviolet. By using the quadrupole spectrum it was possible to obtain greatly improved rotational and vibrational constants of the hydrogen molecule in its ground state. These data are of particular interest from the point of view of a calculation of the thermodynamic properties of hydrogen.

Unless the amounts of hydrogen present in the atmospheres of the major planets are very much larger than those used in the laboratory, spectra of very high resolution will be required in order to detect hydrogen by means of these quadrupole lines in planetary spectra. Such high resolution has not yet been accomplished. But Kuiper [7], at Yerkes Observatory, has observed a diffuse feature in the spectra of Uranus and Neptune at 8267Å which is exactly at the position of the line $S(0)$ of the 3—0 quadrupole band. This line would be the strongest line expected at very low temperature. How-

ever, the considerable width of the planetary feature (about 40Å) makes it impossible to identify it as the quadrupole line.

More recently the rotation-vibration spectra of hydrogen as well as of other homonuclear molecules have been observed by Crawford, Welsh, Locke, and their collaborators [8; 9] at Toronto in absorption *at high pressure*, i.e., as forbidden transitions of type 2(*b*). These spectra are very different from the quadrupole spectrum in that their intensity increases with the square of the pressure and that in place of sharp lines very broad features (about 300 cm^{-1} wide) are observed.

Condon [10], in 1932, predicted a rotation-vibration spectrum of homonuclear molecules induced by a constant electric field. This spectrum can also be considered as a Raman spectrum at zero exciting frequency. The same three branches are expected as occur in the quadrupole spectrum. It appears that the spectra observed by Crawford and Welsh correspond to this spectrum except that the external electric field is replaced by the collisions with other molecules.

Crawford and Welsh observed that the *line width* of this pressure-induced spectrum decreases with decreasing temperature. This is shown for the 2—0 band in Fig. 5. While for the fundamental (1—0 band) at low temperature essentially three lines occur, $Q(1)$, $S(0)$,

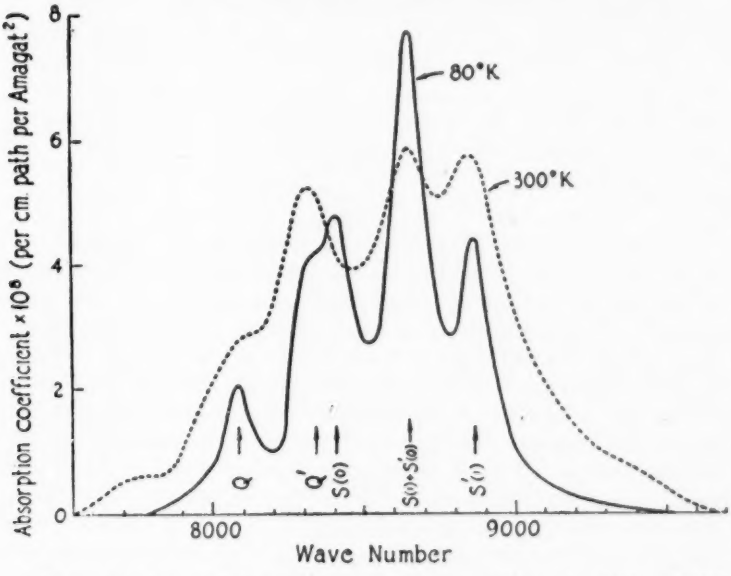

FIGURE 5.—Induced infrared absorption of H_2 near 8500 cm^{-1} at 300° and 80°K according to Welsh, Crawford, MacDonald, and Chisholm [9].

and $S(1)$, for the overtone (2—0 band) some additional lines, marked Q', $S'(0)$, $S'(1)$ appear (see below).

Since the positions of the pressure-induced lines of the 1—0 and 2—0 bands agree closely with those of the quadrupole lines it appeared probable [11] that the planetary feature observed by Kuiper at 8267Å whose position agrees with $S(0)$ of the 3—0 quadrupole band might be the $S(0)$ line of the pressure-induced 3—0 band. A test of this assumption was therefore made with the high pressure, low temperature absorption tube mentioned in the introduction [12]. In agreement with the results of Welsh and Crawford, at room temperature the 3—0 band was found to consist of very diffuse features. When, however, the temperature was lowered to that of liquid nitrogen a much clearer and more definite spectrum was obtained which is shown in Fig. 6*a*. In this spectrogram actually more lines are seen than

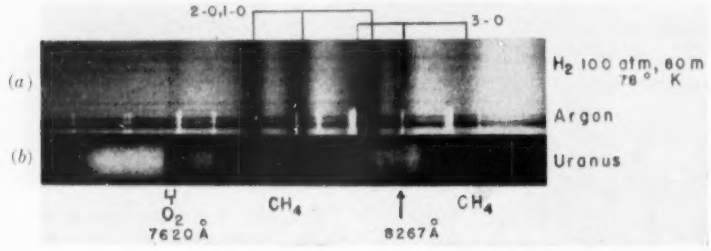

FIGURE 6.—Induced infrared absorption of H_2 near 8270Å observed (*a*) in the laboratory and (*b*) in the spectrum of Uranus after Kuiper. The three diffuse lines marked 3—0 are $Q(1)$, $S(0)$, and $S(1)$ (from right to left) of the single transition 3—0. The three lines marked 2—0, 1—0 are $Q(1)$, $S(0)$, and $S(1)$ of the double transition.

expected. The expected lines are the three lines at the right, i.e., $Q(1)$, $S(0)$, and $S(1)$. The two lines at shorter wavelengths cannot be accounted for as lines of the ordinary pressure-induced spectrum. Similar additional lines occur in the 2—0 band (Fig. 5) although they are more overlapped by the normal lines. They have been explained by Welsh and Crawford as due to a *double transition*, that is, a transition in which each of the two colliding molecules goes from the lowest vibrational level to that with $v = 1$. In other words, the wave number of this transition is twice the wave number of the 1—0 band with rotational energy added in the same way as for the single (2—0) transition. Similarly the extra lines in the region of the 3—0 band (which are actually stronger than the lines of the single transition) can be accounted for as due to a double transition in which one of the

collision partners goes from $v = 0$ to 1, the other simultaneously from $v = 0$ to 2. Table I shows the wavelengths predicted on this basis as well as those of the observed lines. The central line has to be assigned to both the single band and the double band. The agreement between observed and calculated lines is so good that hardly a doubt

TABLE I

OBSERVED AND PREDICTED PRESSURE-INDUCED SPECTRUM OF H_2

Designation		Observed λ (Å)	Predicted λ (Å)
Single transition (3—0)	$Q(0)$	8469	8488
	$Q(1)$		8497
	$S(0)$	8258	8273
	$S(1)$		8151
Double transition (2—0, 1—0)		8166	
	$Q(0)$		8165
	$Q(1)$		8171
	$S(0)$	7958	7956
	$S(1)$	7830	7832

can remain as to the correctness of the interpretation of the spectrum as due to a single and a double transition even though it appears rather difficult to account on a theoretical basis for the occurrence of such a double transition.

It appears to be significant that the line $S(0)$ of the single transition agrees closely *both in wavelength and in width* with the feature 8267Å observed in the spectra of Uranus and Neptune. Fig. 6*b* shows, for comparison with Fig. 6*a*, one of Kuiper's Uranus spectra on which the 8267Å feature can be seen. With one exception the other lines of the H_2 spectrum are obscured in the planetary spectra by strong CH_4 bands. The line 8166Å is not completely obscured by CH_4 and there is indeed evidence that it is present in the Uranus spectrum. However, it appears to be considerably weaker relative to 8267Å than in the laboratory spectrum. This difference can be accounted for by the assumption of the presence of large amounts of helium in addition to hydrogen, since

collisions of He and H_2 would enhance the single transition but not the double transition. At any rate it appears that the presence of large amounts of H_2 in the atmospheres of Uranus and Neptune is now established by means of the pressure-induced H_2 spectrum. The amount of H_2 necessary to account for the observed intensity of the 8267Å feature in Uranus can be roughly obtained by comparison with the laboratory data on the basis of the quadratic dependence on the pressure. Assuming a temperature of 78°K, a helium-hydrogen ratio of 3 : 1, and therefore a scale height of 18 km., one finds a partial pressure of 2 atm. of H_2 at the bottom of the visible atmosphere.

Another example of a forbidden rotation-vibration spectrum is the *rotation-vibration spectrum of* HD. In its equilibrium position the HD molecule, just as the H_2 molecule, has no dipole moment and therefore no rotation-vibration spectrum would be expected. This spectrum is also forbidden by the selection rule that even electronic states combine only with odd ones, a rule that applies to all molecules whose nuclei have equal charge. However, when the nuclei are vibrating, as shown in Fig. 7, since the centre of mass remains at rest but is no longer symmetrically placed, both the centre of the "positive" and that of the "negative" charges move back and forth. On account of the inertia of the electrons the centre of the "negative" charges will lag behind that of the "positive" charges, and therefore an oscillating dipole moment arises which in turn produces a rotation-vibration spectrum. Wick [13] about fifteen years ago calculated the intensity of the 1—0 band of this spectrum. According to his formulae it

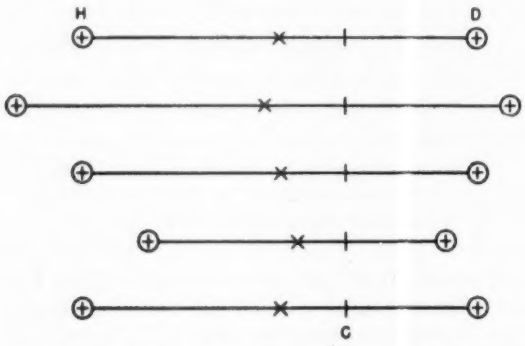

FIGURE 7.—Successive stages of vibration of HD showing origin of dipole moment. The centre of mass is at *C*, the centre of both positive and negative charges would be at × if the inertia of the electrons were neglected.

would require an absorbing path of the order of 30 m. atm. to detect this band. With the absorption tube described earlier it has been possible to observe the 3—0 and 4—0 bands of HD at 9650 and 7420Å using an absorbing path of 1000 m. at a pressure of 1 atm. [14]. Fig. 8 shows the 4—0 band. Unlike the quadrupole spectrum of hydrogen, in the present case one has the usual simple structure of a *P* and an *R* branch which are clearly visible. By an exact measurement of these bands it has been possible to obtain improved molecular constants of the HD molecule.

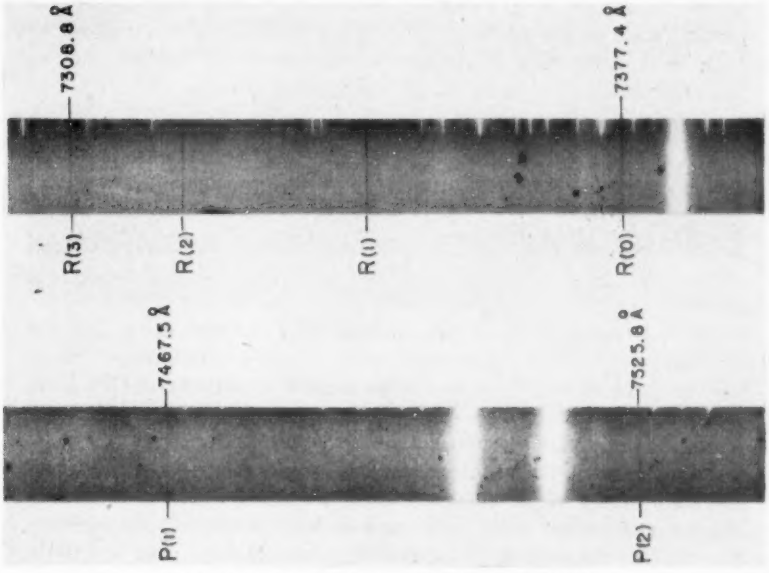

FIGURE 8.— 4—0 rotation-vibration band of HD.

A surprising result is the comparatively small difference in intensity of the 3—0 and 4—0 bands of HD contrary to the usual very rapid decrease of intensity in the series of overtones. This small intensity decrease in the case of HD is apparently connected with the fact that the dipole moment is produced by the vibration. Quantum-theoretical calculations by Wu [15] at Ottawa have indeed shown quantitatively that such a small intensity decrease should be expected. It will be interesting to observe the pressure-induced spectrum of HD, but such experiments have not yet been carried out.

E. Forbidden Electronic Band Spectra

Let us turn now to the discussion of forbidden electronic band spectra, that is, transitions in which the molecule goes from one electronic state to another such that one or more of the electronic selection rules are violated. The most common type of forbidden electronic transition in diatomic molecules is the *intercombination between two states of different multiplicity* (that is, a transition violating the approximate selection rule $\Delta S = 0$). The first case of this type that was observed and identified as such was the *Cameron system of CO* which represents a $^3\Pi - {}^1\Sigma$ transition, the lower state being the ground state of the molecule [16]. These bands have been observed both in emission and in absorption. In absorption they require an absorbing path of about 1 m. at atmospheric pressure. Fig. 9 shows

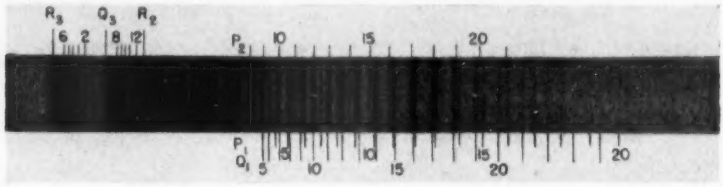

Figure 9.— 0—0 Cameron band of CO in absorption.

the structure of one of the bands as obtained in absorption [17]. Each band consists of three overlapping components of three branches each, corresponding to the three components of the $^3\Pi$ state.

Transitions similar to the Cameron bands of CO have been observed for many of the halides of the elements of the third column of the periodic system by Miescher and Wehrli [18]. At Ottawa recently, Sharma [19] observed the corresponding transitions in AlCl and AlBr. In these cases the triplet splitting in the $^3\Pi$ state is very large compared to the rotational constant B, and therefore a coupling case applies in which only two of the three components of the $^3\Pi$ state combine with the ground state. These are the transitions $^3\Pi_1 - {}^1\Sigma$ and $^3\Pi_0 - {}^1\Sigma$ of which the former gives rise to bands with P, Q, and R branches, whereas the latter gives bands with P and R branches only.

Another important case is the so-called *Vegard-Kaplan bands of nitrogen* which represent a $^3\Sigma_u^+ - {}^1\Sigma_g^+$ transition, the lower state being the ground state of the N_2 molecule. These bands have been observed in emission in certain electric discharges [20; 21; 22; 23] as well as in the spectrum of the night sky [24; 25]. One of the first objects of my own work on absorption spectra with long absorbing paths was the

observation of the Vegard-Kaplan bands of nitrogen in absorption in order to verify their assignment as a $^3\Sigma_u^+ - {}^1\Sigma_g^+$ transition and also in order to determine their absolute intensity. These absorption bands are expected to be in the vacuum region and it is not easy to be sufficiently free from the much stronger absorption due to oxygen in order to observe them. Recently with our new vacuum spectrograph at Ottawa we have tried again to observe these bands in absorption, and, using very pure nitrogen, have been able to ascertain that with an absorbing path of 6 m. at atmospheric pressure they do not occur. Apparently the transition probability for this transition is very much lower than that of the Cameron bands of CO. The difference between the two cases seems to lie in the fact [26] that in the case of CO there is a state, $^1\Pi$, of the same electron configuration which combines strongly with the ground state and from which the forbidden $^3\Pi - {}^1\Sigma$ transition can borrow intensity, while in the case of N_2 the excited electron configuration does not give rise to a state that combines strongly with the ground state. Another example of this type is an extremely weak absorption system of oxygen only recently discovered among the forbidden $\Sigma^+ - \Sigma^-$ bands to be discussed later. This transition is of the type $^1\Sigma_u^- - {}^3\Sigma_g^-$.

We have tried to find intercombinations in a number of other diatomic molecules: NO, for which a $^4\Pi - {}^2\Pi$ is expected, SiO, for which the analogue of the Cameron bands of CO, $^3\Pi - {}^1\Sigma$, is expected; NH, for which a $^1\Sigma - {}^3\Pi$ transition is expected; and CH, for which various intercombinations involving the low lying but unobserved $^4\Sigma^-$ state are expected. In all these cases our search has thus far been unsuccessful. Further search with improved means would seem important in order to round out our knowledge of these molecules.

While the intercombinations are forbidden transitions of type 1, there are also a number of known cases of forbidden transitions of type 2(*a*), that is, transitions rigorously forbidden as electric dipole radiation but actually occurring weakly on account of the possibility of *magnetic dipole radiation*. The best known case of this type is the *red atmospheric oxygen bands* which, as was shown by Mulliken [27] and Van Vleck [28], represent a $^1\Sigma_g^+ - {}^3\Sigma_g^-$ transition. The structure of the bands is explained in Fig. 10. Since the oxygen nucleus has zero spin only the symmetric rotational levels occur which are, in the upper state, the levels with even K, and in the lower state, those with odd K. On account of the selection rule $\Delta J = 0, \pm 1$, each one of the upper state rotational levels can combine with only two of the three triplet components of a given lower triplet, thus giving rise to two series of apparent doublets.

Another example whose structure is also explained in Fig. 10 is the *infrared atmospheric oxygen bands* which represent a $^1\Delta_g - {}^3\Sigma_g^-$ transition. In this case rotational levels occur for each J value of the upper state $^1\Delta_g$ and, therefore, additional branches occur, one each in which the rotational quantum number K changes by ± 2, and three branches in which it changes by 0. All these branches have been observed [29] in the 0—0 band which occurs prominently in the solar spectrum at

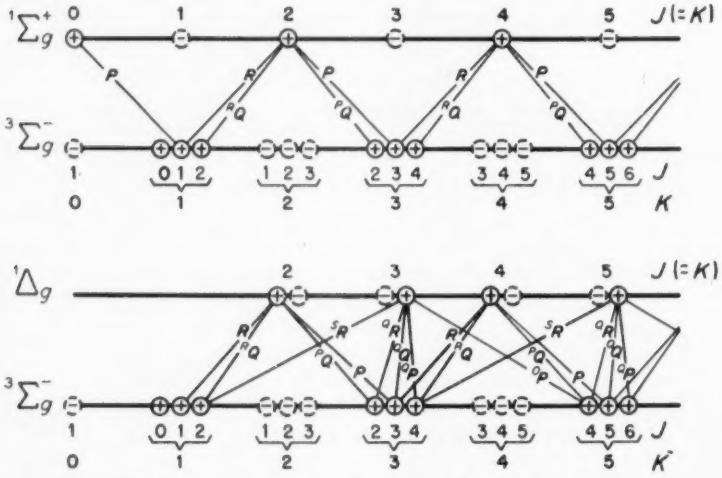

FIGURE 10.—Origin of branches in $^1\Sigma_g^+ - {}^3\Sigma_g^-$ and $^1\Delta_g - {}^3\Sigma_g^-$ bands.

12700Å. While the red atmospheric oxygen bands can be obtained in absorption with a path length of 6 m. at atmospheric pressure, the infrared bands require a much longer path of the order of 50 m. The reason for this difference is clear when it is considered that the infrared bands, in addition to being a magnetic dipole intercombination transition, violate the rule $\Delta \Lambda = 0, \pm 1$. The red atmospheric oxygen bands have also been observed in emission, both in the spectrum of the night sky [30] and in the laboratory in certain flames and afterglows [31; 32].

Still another example of a magnetic dipole transition is the system of far ultraviolet absorption bands of the nitrogen molecule usually referred to as *Lyman-Birge-Hopfield bands* [33]. They extend from 1450Å to shorter wavelengths and represent the first moderately strong absorption system of the N_2 molecule. In order to observe this system an absorbing path of about 40 cm. at atmospheric pressure is required.

It was recognized only a few years ago [34] that these bands represent a magnetic dipole transition of the type $^1\Pi_g - {}^1\Sigma_g^+$. At Ottawa we have recently re-investigated the Lyman-Birge-Hopfield bands [35] in emission in an electric discharge at low pressure and have established the presence of a predissociation in the upper level, at an energy of about 9.7 e.v. This predissociation is of some importance because it will lead to the production of free nitrogen atoms in the upper atmosphere by solar radiation below 1270Å.

The first long absorption tube that I built over twenty years ago gave an absorbing path of 25 m. When it was filled with oxygen at atmospheric pressure, I found a new very weak absorption band system of oxygen in the region 2500Å [36]. The bands of this system appear under medium dispersion to consist only of Q branches and were therefore ascribed to a $\Sigma^+ - \Sigma^-$ transition, i.e., a forbidden transition of type 1. In the present case the approximation to which the selection rule ($\Sigma^+ \leftarrow\!|\!\rightarrow \Sigma^-$) holds is extremely good, and these bands are even weaker than magnetic dipole transitions. More recently [37] this band system has been re-investigated under high dispersion with the much longer absorbing paths set up at Yerkes Observatory. More bands at longer wavelengths have been found and the fine structure has been completely resolved. Fig. 11 shows part of one of the bands under high resolution. It is seen that each of the "lines" of the Q branch is resolved into six component lines and, in addition, some fainter lines belonging to different branches are present. The fine structure is in

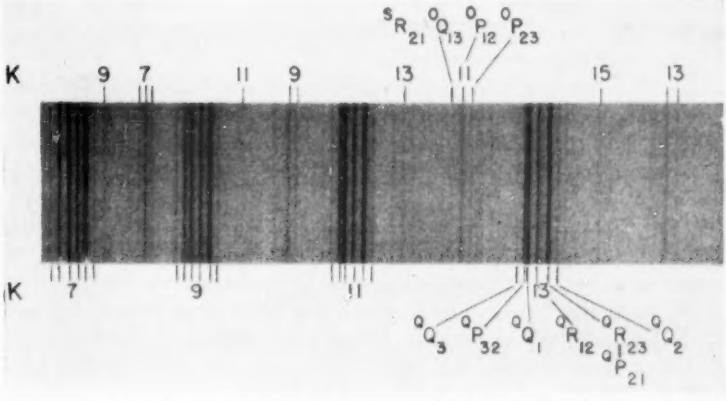

FIGURE 11.—Part of the 9—0 band of the $^3\Sigma_u^+ - {}^3\Sigma_g^-$ system of O_2 under high resolution. Under low dispersion the groups of lines marked $K = 7, 9, 11, 13$ below form one apparently single Q branch.

agreement with the assumption that this transition is a $^3\Sigma_u^+ - {}^3\Sigma_g^-$ transition which, although forbidden by the selection rule $\Sigma^+ \leftarrow|\rightarrow \Sigma^-$, occurs weakly on account of spin-orbit interaction. Fig. 12 gives an energy level diagram for such a transition. It is found that the triplet splitting in the upper state is about twice as large as in the ground state of O_2 and has the opposite sign. Applying the selection rule $\Delta J = 0, \pm 1$, it is readily seen from this diagram how seven Q branches arise, two of which coincide, and how, in addition, branches with $\Delta K = \pm 2$ are to be expected, i.e., O and S form branches. The observed relative intensities of the various branches do not agree well with those predicted from theory. However, on account of the closeness to the dissociation limit, special coupling conditions apply which were not considered in the theoretical prediction. It may be of interest to recall that this band system has been identified as occurring in emission in the light of the night sky [38; 25].

The $^3\Sigma_u^+ - {}^3\Sigma_g^-$ bands observed in absorption form a simple progression with a clear convergence limit at 2425Å. The energy of this dissociation limit is slightly higher than expected from the well-known dissociation limit in the Schumann-Runge bands of oxygen by subtracting the excitation energy of the ^1D state of atomic oxygen. The reason for this discrepancy may lie in the fact that the dissociation products of the $^3\Sigma_u^+$ state are not $^3P_2 + {}^3P_2$, but $^3P_2 + {}^3P_{1 \text{ or } 0}$. However, if that were the case, a contradiction to the non-crossing rule would arise and it is perhaps more likely that the fault lies with the slight extrapolation made in obtaining the dissociation limit of the Schumann-Runge bands. If this were correct the dissociation energy of the O_2 molecule would have to be changed from the value 5.080 e.v. now usually quoted to 5.109 e.v.

Finally, I should like to consider examples of forbidden electronic transitions of type 2(b) which have also been observed for the oxygen molecule. If the absorption spectrum of oxygen is studied *at higher pressures*, a number of completely diffuse bands are observed in the visible region in addition to the ordinary atmospheric oxygen bands. The strongest of these bands occur at 6300, 5780, 4775, 4475, 3815, and 3617Å. Even under high resolution there is no indication of any fine structure. These bands can be observed at comparatively low pressures if the path is sufficiently long. The intensity of these bands increases with the square of the pressure and they have generally been ascribed to the presence in oxygen of O_4 molecules. As was first shown by Ellis and Kneser [39], all these bands can be well accounted for, as far as their wavelengths are concerned, as due to *double electronic transitions* in an O_4 molecule or quasi molecule. If in both O_2 molecules forming

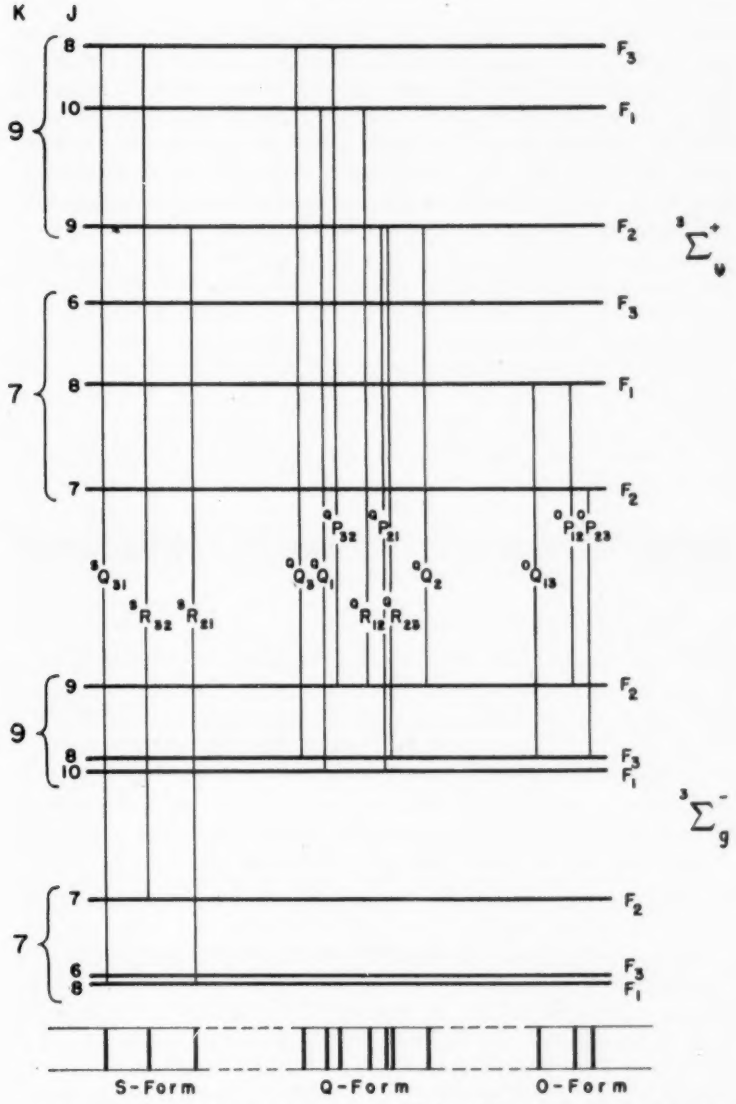

FIGURE 12.—Energy level diagram for a $^3\Sigma_u^+ - {}^3\Sigma_g^-$ band. The strip at the bottom gives schematically the resulting spectrum. Only one line of each branch is shown.

an O_4 molecule a transition to the $^1\Delta$ state takes place simultaneously, the resulting wavelength is 6342Å which agrees with one of the strong diffuse bands. Similarly, if in one of the O_2 molecules a transition to the $^1\Delta$ state takes place, and in the other a transition to the $^1\Sigma$ state, the resulting wavelength is 4760Å; and if in both O_2 molecules the transition to $^1\Sigma$ takes place the wavelength is 3810Å. Both these wavelengths again agree with two of the observed diffuse bands. The remaining bands can be accounted for by adding vibrational quanta of the $^1\Delta$ and $^1\Sigma$ state. It would appear that there is a close analogy of these double transitions of oxygen to the double transitions in the rotation-vibration spectrum of hydrogen discussed previously.

In addition to the diffuse bands in the visible region, there occurs at high pressure a progression of ultraviolet bands which almost coincide with the $\Sigma^+ - \Sigma^-$ bands of oxygen [40; 41]. However, these pressure-induced bands may be observed simultaneously with the $\Sigma^+ - \Sigma^-$ bands and cannot, therefore, be just pressure-broadened $\Sigma^+ - \Sigma^-$ bands. Fig. 13 shows a spectrum in which both systems can be seen.

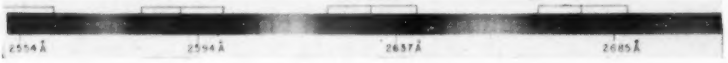

FIGURE 13.—Near ultraviolet absorption of O_2 at 8 atm. pressure, and a path length of 90 m. The discrete $^3\Sigma_u^+ - {}^3\Sigma_g^-$ bands are marked at the bottom, the diffuse triplet bands of Wulf and Finkelnburg and Steiner are marked at the top.

Each of the pressure-induced bands has three maxima, but otherwise is completely diffuse. With the long paths used in the more recent work it has been possible to observe at lower pressures a system of very faint and rather complicated bands which may be the exact analogue in free O_2 of the pressure-induced bands. These bands have tentatively been assigned to a $^3\Delta_u - {}^3\Sigma_g^-$ transition in oxygen.

F. CONCLUSION

Just as for diatomic molecules, forbidden transitions are of considerable importance for the study of polyatomic molecules. Singlet-triplet intercombinations, as well as other forbidden transitions of type 1, have been found for them (for example, the well-known near-ultraviolet absorption bands of C_6H_6). As far as I am aware, no forbidden transitions of type 2(*a*), i.e., magnetic dipole and quadrupole transitions, are known for polyatomic molecules. Pressure-induced infrared bands of CO_2, i.e., forbidden transitions of type 2(*b*), have been observed by Welsh and Crawford [8]. Both for polyatomic and

for diatomic molecules a good deal of work remains to be done on forbidden transitions. Experimentally, the discovery of new forbidden transitions and the study of their intensities and of the conditions of their occurrence will contribute to a better understanding of the structure of the molecules considered and to the solution of certain astrophysical and chemical problems. Further theoretical work on forbidden transitions is needed in order to understand such phenomena as the occurrence of double transitions in collisions between two hydrogen molecules and two oxygen molecules, the intensity distribution in the branches of forbidden bands, and the absolute intensity of forbidden transitions.

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TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA
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SECTION THREE

Dominated Convergence of Kantorovitch Polynomials in the Space L^p

P. L. BUTZER

Presented by R. L. JEFFERY, F.R.S.C.

1. Introduction

If $f(x)$ belongs to the class L^1, of Lebesque integrable functions over the closed interval $[0, 1]$, then the Kantorovitch polynomial (see [3] or [4]) corresponding to the function $f(x)$ defined in $[0, 1]$ is

$$P_n{}^f(x) = \sum_{\nu=0}^{n} (n+1)p_{\nu,n}(x) \int_{\nu/(n+1)}^{\nu+1/(n+1)} f(t)\,dt, \tag{1}$$

where

$$p_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

In the form of a singular integral, this polynomial can be written

$$P_n{}^f(x) = \int_0^1 K_n(x,t) f(t)\,dt \tag{2}$$

with non-negative kernel

$$K_n(x,t) = (n+1)p_{\nu,n}(x) \tag{3}$$

for $\dfrac{\nu}{n+1} < t \leqslant \dfrac{\nu+1}{n+1}$, $\nu = 0, 1, 2, \ldots, n$; $K_n(x,0) = 0$; $0 \leqslant x \leqslant 1$.

Let

$$F(x) = \int_0^x f(t)\,dt.$$

At every point x, where $f(x) = F'(x)$, that is, almost everywhere in $[0, 1]$, Lorentz [4] has proved that

$$\lim_{n\to\infty} P_n{}^f(x) = f(x). \tag{4}$$

In §2 we obtain an independent proof of this result as an immediate application of a general theorem. In §3 we shall show that the poly-

nomials $P_n f(x)$ approach the corresponding function $f(x)$ dominatedly (for this term, in the space L^1, see [2, p. 6]) in the space L^p, $p > 1$. As a corollary of this fundamental theorem we deduce that the $P_n f(x)$ "converge strongly" to $f(x)$; the proof of the latter result previously was based on a rather particular theorem (cf. [4]).

I with to thank Professor G. G. Lorentz and Professor W. J. Webber for their kind interest and helpful suggestions. This paper was written during the tenure of a grant from the National Research Council, Ottawa.

2. Convergence of the $P_n f(x)$

Lemma 1. *If the non-negative kernel $\chi_n(x, t)$, $0 \leqslant x \leqslant 1$, as a function of t increases for $t < x$ and decreases for $t > x$, then the conditions*

$$\lim_{n\to\infty} \int_0^1 \chi_n(x, t)\, dt = 1, \tag{5}$$

$$\chi_n(x, t) \to 0 \text{ as } n \to \infty,\ x \neq t, \tag{6}$$

imply that

$$\int_0^1 \chi_n(x, t) f(t)\, dt \to f(x) \tag{7}$$

at all points where $f(x)$ is the derivative of its indefinite integral.

For the proof, we refer the reader to the work of Romanovsky [6].

We shall now show that our kernel (3) satisfies the hypothesis of the lemma. Obviously

$$\int_0^1 K_n(x, t)\, dt = \sum_{\nu=0}^{n} p_{\nu,n}(x) = 1. \tag{8}$$

Now if $\delta = n^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, then it is known that (cf. [4]) for every $k > 0$ there exists a constant C depending only on α and k such that

$$\sum_{|\nu/n - x| \geqslant \delta} p_{\nu,n}(x) \leqslant Cn^{-k}.$$

From this inequality we obtain $(n + 1)p_{\nu,n}(x) \to 0$ for $|\nu/n - x| \geqslant \delta > 0$, so $K_n(x, t) \to 0$ for $t \neq x$. Furthermore as

$$\frac{p_{\nu,n}(x)}{p_{\nu-1,n}(x)} = \frac{x}{1 - x} \cdot \frac{n - \nu + 1}{\nu},$$

$p_{\nu,n}(x)$ as a function of ν increases with ν for fixed n and x as long as $\nu/n + 1 \leqslant x$ and then decreases for larger ν. Hence $K_n(x, t)$ increases for $t < x$ and decreases for $t > x$. From (7) we obtain $P_n f(x) \to f(x)$ p.p. (that is, almost everywhere).

3. Dominated Convergence of the $P_n^r(x)$ in the Space L^p

First of all, we need the inequality

$$(9)\quad \sum_{\nu=0}^{n-1}(n+1)\left|\frac{\nu+1}{n+1}-x\right|\left|p_{\nu,n}(x)-p_{\nu+1,n}(x)\right| \leqslant 1, \qquad 0 \leqslant x \leqslant 1.$$

Since

$$p_{\nu,n}(x)-p_{\nu+1,n}(x)=p_{\nu,n}(x)\frac{(\nu+1)-(n+1)x}{(\nu+1)(1-x)}$$

$$=\frac{1}{x}\frac{\nu+1}{n+1}\cdot\frac{(\nu+1)-(n+1)x}{(\nu+1)(1-x)}\cdot p_{\nu+1,n+1}(x),$$

the left-hand side of (9) equals

$$\frac{1}{x(1-x)}\sum_{\nu=0}^{n-1}\left|\frac{(\nu+1)-(n+1)x}{n+1}\right|\left|\frac{(\nu+1)-(n+1)x}{\nu+1}\right|$$

$$\cdot(\nu+1)p_{\nu+1,n+1}(x)$$

$$=\frac{1}{x(1-x)}\sum_{\nu=0}^{n-1}[(\nu+1)-(n+1)x]^2\frac{p_{\nu+1,n+1}(x)}{n+1}\leqslant 1$$

because it is known that (see [4] or [7], p. 152])

$$\sum_{\nu=0}^{n}(\nu-nx)^2p_{\nu,n}(x)=nx(1-x).$$

This establishes the inequality (9).

We shall also make use of the following general lemma.

Lemma 2. *Let $\chi_n(x,t)$, fixed $0 \leqslant x \leqslant 1$ be a non-negative function of bounded variation in the variable t, $0 \leqslant t \leqslant 1$, depending on the parameter n, with $\chi_n(x,0)=0$ and satisfying the conditions*

$$(10)\quad \int_0^1 \chi_n(x,t)\,dt \leqslant C,$$

$$(11)\quad \int_0^1 |t-x|\cdot|d_t\chi_n(x,t)| \leqslant C_1,$$

where C and C_1 are independent of n. Define

$$H_n(x)=\int_0^1 \chi_n(x,t)f(t)\,dt$$

where $f(t) \in L^1[0,1)$. Then

$$\sup|H_n(x)| \leqslant A\cdot\theta(x;f) \text{ p.p.},$$

where the constant $A(\equiv C+2C_1)$ is independent of f and n and

$$\theta(x;f) = \sup_{\substack{0 \leqslant t \leqslant 1 \\ t \neq x}} \frac{1}{t-x} \int_x^t |f(u)|\, du.$$

The proof of this lemma follows by a method similar to the one in [8, pp. 246–7].

The hypothesis of this lemma is readily seen to be satisfied by our kernel $K_n(x, t)$. In particular, the relation (8) shows that the inequality (10) is satisfied with $C = 1$ and on account of (9), (11) is satisfied with $C_1 = 1$. Hence we obtain that the polynomials $P_n{}^f(x)$ are majorized by the function $3\theta(x; f)$ for every n $(n = 0, 1, 2, \ldots)$, that is,

$$\sup_n |P_n{}^f(x)| \leqslant 3\,\theta(x;f). \tag{12}$$

Now Hardy and Littlewood (see [8, p. 244]) have shown that if $f \in L^p[0, 1]$, $p > 1$ then $\theta(f) \in L^p[0, 1]$ with

$$\int_0^1 \theta^p(x;f)\, dx \leqslant 2\left(\frac{p}{p-1}\right)^p \int_0^1 |f(x)|^p\, dx. \tag{13}$$

In view of the relations (4), (12), and (13), we deduce the fundamental theorem.

THEOREM. *If $f \in L^p[0, 1]$, $p > 1$, the polynomials $P_n{}^f(x)$ are dominatedly convergent almost everywhere in the space L^p, in other words*

(i) $P_n{}^f(x) \to f(x)$ p.p. *in* [0, 1],

(ii) *there is an element $\theta(x; f)$ in the space L^p such that $|P_n{}^f(x)| \leqslant 3\theta(x;f)$* p.p., *for all n.*

COROLLARY. *If $f \in L^p[0, 1]$, $p > 1$, the polynomials $P_n{}^f(x)$ converge strongly to $f(x)$, namely*

$$||P_n{}^f - f||_{L^p} \equiv \left[\int_0^1 |P_n{}^f(x) - f(x)|^p\, dx\right]^{1/p} \to 0. \tag{14}$$

For, $|P_n{}^f(x) - f(x)|^p$ approaches zero almost everywhere and is dominated by an integrable function, and hence by Lebesgue's general convergence theorem the relation (14) follows.

The same methods of proof have been used to establish similar theorems in the Banach spaces $\Lambda(\alpha, p)$ and $M(\alpha, p)$, $0 < \alpha < 1$, $p > 1$, introduced by Lorentz [5]. The details are found in [1].

Recently, E. Levi [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8), vol. 9 (1950), 242-246; (8), vol. 10 (1951), 360-364] has also established the above corollary proving that the $P_n{}^f(x)$ converge strongly to $f(x)$ in the space L^p, $p \geqslant 1$. His proof involves approximations by step functions.

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SECTION THREE

On the Relation between the P^2-integral and the Cesàro-Perron Scale of Integrals

H. W. ELLIS

Presented by R. L. JEFFERY, F.R.S.C.

1. R. D. James [3] has defined a second Perron integral (P^2-integral) and shown that it contains and is more general than the Cesàro-Perron integral of J. C. Burkill. The Cesàro-Perron integral is the integral of order one in the Cesàro-Perron scale [1] of integrals of increasing generality. The purpose of this note is to give examples showing that there are functions that are P^2-integrable without being C_rP-integrable for any r and that, for r an integer greater than one, there are functions that are C_rP-integrable without being P^2-integrable.

2. *The functions*

$$f(p, x) = x^{-p} \sin x^{-q},\ g(p, x) = x^{-p} \cos x^{-q}, \quad p > 0,\ q > 0,\ x \neq 0;$$

$$f(p, 0) = g(p, 0) = 0$$

are C_rP-integrable on any finite interval containing the origin if and only if $r > (p - q - 1)/q$.

Fix $2k > (p - q - 1)/q$. The $2k$th derivative $f^{(2k)}[p - 2k(q + 1), x]$ exists for $x \neq 0$ and consists of a linear combination of the functions $f(p, x)$, $g(p - q, x)$, . . . , $f[p - 2kq, x]$. Defining

$$F_r(x) = f^{(2k-r)}[p - 2k(q + 1), x], \qquad 0 \leqslant r \leqslant 2k,\ x \neq 0;$$

$$F_r(0) = 0, \qquad f(x) = F_0(x)$$

gives

$$F_{r+1}(x) = 0[x^{(r+1)(q+1)-p}] \qquad \text{as } x \to 0. \tag{1}$$

The function $f(x)$ will be C_rP-integrable on (a, b) (with $F_1(x) - F_1(a)$ serving as both a major and minor function) if $F_1(x)$ is C_r-continuous at $x = 0$, that is if

$$\lim_{x \to 0} \frac{r}{x^r} \int_0^x (x - t)^{r-1} F_1(t)\, dt = F_1(0) = 0,$$

the integration being in the $C_{r-1}P$-sense. By successive integrations by parts, valid for the Cesàro-Perron integrals involved, this is equivalent to

$$F_{r+1}(x) = 0(x^r) \qquad as\ x \to 0. \tag{2}$$

It follows that $f(x)$ is C_rP-integrable on (a, b) if $r > (p - q - 1)/q$.

Similarly $f_1(x) = g^{(2k)}[p - 2k(q + 1) - q, x]$, $x \neq 0$, $f_1(0) = 0$ is a linear combination of the functions $g(p - q, x)$, $f(p - 2q, x)$, . . . , $g[p - (2k + 1)q, x]$ and is C_rP-integrable on (a, b). Continuing in this way a C_rP-integrable linear combination $f_n(x)$ of the functions $f(p - nq, x)$, . . . , $f[p - (2k + n)q, x]$ is obtained for n even and a similar combination with f and g interchanged for n odd. For the smallest integer $n > (p + 1)/q$, all of the terms of $f_n(x)$ except $f(p - nq, x)$ (n even) or $g(p - nq, x)$ (n odd) will be Lebesgue integrable and therefore $f(p - nq, x)$ or $g(p - nq, x)$ will be C_rP-integrable. Reversing the order in which the above linear combinations were formed, the C_rP-integrability of the function of lowest order follows at each step and finally the C_rP-integrability of $f(p, x)$ is obtained.

A consideration of (1) and (2) shows that $f(x)$ is not C_rP-integrable if $r \leqslant (p - q - 1)/q$. We may suppose that $r + 1 > (p - q - 1)/q$, for if it is shown that $f(p, x)$ is not C_rP-integrable for such a value of r it follows from the consistency of the Cesàro-Perron scale of integrals that $f(p, x)$ cannot be $C_{r'}P$-integrable for $r' < r$. Under this hypothesis $f_1(x)$ is C_rP-integrable as are all of the terms of $f(x)$ except $f(p,x)$. It follows that $f(p, x)$ cannot be C_rP-integrable on (a, b).

3. *For r fixed there are functions that are P^2-integrable without being C_rP-integrable.* Consider

$$F(x) = x^\alpha \sin x^{-\beta},\ G(x) = x^\alpha \cos x^{-\beta},\ x \neq 0,\ F(0) = G(0) = 0,$$

$$f(x) = F''(x),\ g(x) = G''(x),\ x \neq 0,\ f(0) = g(0) = 0.$$

By James's Theorem 4.2 [3] the functions $f(x)$, $g(x)$ will be P^2-integrable over (a, c, b) where (a, b) contains the origin if $F(x)$, $G(x)$ are continuous and smooth at $x = 0$. Continuity will be assured by (i) $\alpha > 0$; smoothness by (ii) x^β even, x^α odd for $F(x)$ or $G(x)$, or x^β odd, x^α even for $F(x)$ or both odd for $G(x)$. From §2, $f(x)$ will not be C_rP-integrable if (iii) $\alpha \leqslant \beta(1 - r) + 1$. Values of α and β satisfying (i)–(iii) include $\beta = 2(2r + 1)^{-1}$, $\alpha = 3(2r + 1)^{-1}$ in the definition of $F(x)$ or $G(x)$, or $\beta = (2r + 1)^{-1}$, $\alpha = (r + 2)/(2r + 1)$ in the definition of $F(x)$ if r is even and of $G(x)$ if r is odd.

4. *There are functions that are P^2-integrable without being C_rP-integrable for any r.*

Let I_n denote the interval $[2^{-n}, 2^{-n+1}]$ and define

$$F_n(x) = (x - x_n)^{3/(2n+1)} \cos \frac{C_n}{(x - x_n)^{2/(2n+1)}}; \qquad x, x_n \in I_n,$$

$$F(x) = -F(-x) = \sum_{n=1}^{\infty} a_n F_n(x),$$

$$f(x) = F''(x),\ x \neq 0,\ x \neq x_n;\ n = 1, 2, \ldots;\ f(x) = 0 \text{ elsewhere.}$$

It follows from §3 that $f(x)$ is not C_rP-integrable on $(-1, 1)$ for any r. We shall show that the constants x_n, C_n and $a_n (n = 1, 2, \ldots)$ can be so chosen that $f(x)$ is P^2-integrable over $(-1, c, 1)$ for every $-1 < c < 1$.

$F_n(x)$ has maxima for

$$\left[\frac{C_n}{2k\pi}\right]^{n+\frac{1}{2}} < x - x_n < \left[\frac{2C_n}{(4k-1)n}\right]^{n+\frac{1}{2}}$$

and the values of $k = 1, 2, \ldots$ for which $x \in I_n$. The maximum of $F_n(x)$ corresponding to a fixed k (notation $\max_k F_n(x)$) satisfies

$$\left[\frac{C_n}{2k\pi}\right]^{3/2} < \max_k F_n(x) < \left[\frac{2C_n}{(4k-1)n}\right]^{3/2}.$$

If x_n and c_n are chosen so that $\max_1 F_n(x)$ is at $x = 2^{-n+1}$, $\max_{-p} F_n(x)$ at $x = 2^{-n}$ it may be verified that

$$\frac{\max_{-p} F_n(x)}{\max_1 F_n(x)} < \left[\frac{16}{3(4p-1)}\right]^{3/2}.$$

If $p \geqslant 2$ is fixed and the values a_n chosen so that

$$a_n F_n(2^{-n}) = a_{n+1} F_{n+1}(2^{-n}),$$

it then follows that, since $\max_1 a_{n+1} F_{n+1}(x) = \max_{-p} a_n F_n(x)$,

$$\frac{\max_1 a_{n+1} F_{n+1}(x)}{\max_1 a_n F_n(x)} < \lambda < 1, \text{ all } n.$$

Since $\max_1 a_n F_n(x)$ is the maximum of $F(x)$ on I_n, $F(x)$ is continuous at $x = 0$. Since $F(x)$ is odd and $F(0) = 0$, $F(x)$ is smooth at $x = 0$. By definition $F(x)$ is continuous at the points $\pm 2^{-n}$, $n = 0, 1, 2, \ldots$ and is smooth at these points since both right and left derivatives are zero there. As in §3, $F(x)$ is smooth and continuous at each point x_n and the P^2-integrability of $f(x)$ follows from James's Theorem 4.2.

5. If $\alpha = 0$ in the definition of $g(x)$ in §3, $g(x)$ is not P^2-integrable over $(-1, c, 1)$ since $F(x)$ cannot be defined so as to be continuous at $x = 0$. By §2 $g(x)$ is C_rP-integrable if $r > (\beta + 1)/\beta$ and is therefore C_2P-integrable for $\beta > 1$.

All of the preceding relations hold with the more genreal GM_r-integrals [2] replacing the corresponding C_rP-integrals.

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SECTION THREE

Unbounded Operators and a Theorem of A. Robinson*

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Presented by R. L. JEFFERY, F.R.S.C.

1. Introduction

LET B be a Banach space of elements x with convergence defined as usual by the norm. Let t_{mn} $(m, n = 1, 2, \ldots)$ be linear (not necessarily bounded) operators mapping B into itself. In a recent paper, A. Robinson [4] (see also Melvin-Melvin [3]) has proved the following interesting generalization of the theorem of Toeplitz on regular summability methods. The word "bounded," applied to an operator T (but not to its value Tx) will always mean the existence of the norm $||T||$.

THEOREM 1. *In order that the transformation*

$$y_m = \sum_{n=1}^{\infty} t_{mn}\, x_n \tag{1}$$

be defined for each convergent sequence $x_n \in B$ *and that* $\lim y_m = \lim x_n$ *it is necessary and sufficient that*

$$\lim_{m\to\infty} t_{mn} = 0 \qquad (n = 1, 2, \ldots), \tag{2}$$

$$\sum_{n=1}^{\infty} t_{mn} = t_m \text{ exists for } m = 1, 2, \ldots, \tag{3}$$

$$\lim_{m\to\infty} t_m = I, \tag{4}$$

$$||t_{mN}, t_{m,N+1}, \ldots|| \leqslant M < +\infty \qquad (m = 1, 2, \ldots) \tag{5}$$

for some constants M *and* N.

Here 0, I are the zero and the identical operator, respectively, and (2) means for instance that $t_{mn}\, x \to 0$, $m \to \infty$ for each $x \in B$. The "group norm" $||t_\nu, t_{\nu+1}, \ldots||$ is defined by

$$||t_\nu, t_{\nu+1}, \ldots|| = \sup ||t_\nu\, x_\nu + \ldots + t_\mu\, x_\mu|| \tag{6}$$

taken over all integers $\mu \geqslant \nu$ and all x_p with $||x_p|| \leqslant 1$ $(p = \nu, \ldots, \mu)$.

*This paper was written while the authors were members of the Summer Research Institute, Canadian Mathematical Congress, Kingston, 1952.

Condition (5) implies that the operators t_{mn} must be bounded for $n \geqslant N$. If they are assumed bounded for all n, $m = 1, 2, \ldots$, this theorem follows easily from the Banach-Steinhaus theorem for operators on $c(B)$ (see §2); Robinson and Melvin-Melvin give much longer proofs. The purpose of this note is to give a general theorem on linear (possibly unbounded) operators which seems to be new (Theorem 3 in §3 and its variant, Theorem 4) and to show that the general case of Theorem 1 follows from the special case and this result. The existence of such a theorem was suggested to us by Robinson's general theorem.

2. Bounded Operators

For a given Banach space B, we denote by $c(B)$ the set of all sequences $X = \{x_n\}$, $x_n \in B$, converging in B. $c(B)$ is a Banach space with the norm

$$||X|| = \sup_n ||x_n||.$$

If the t_ν ($\nu = 1, \ldots, n$) are linear operators from B to B, the sum

$$\sum_{\nu=1}^{n} t_\nu x_\nu = TX$$

is a linear operator from $c(B)$ to B; its norm is easily seen to be

$$||T|| = ||t_1, \ldots, t_n|| = \sup_{||x_\nu|| \leqslant 1} ||\sum_1^n t_\nu x_\nu||.$$

For a sequence $t_1, \ldots, t_n, \ldots$, $||t_1, \ldots, t_n|| \to ||t_1, t_2, \ldots||$. A fundamental set in $c(B)$ consists of all sequences of the form $(0, \ldots, 0, x, 0, \ldots)$ and $(x, x, \ldots)$. Hence, by the Banach-Steinhaus theorem [1, p. 80, theorem 5] a series $\sum_1^\infty t_\nu x_\nu$ is a bounded linear operator TX from $c(B)$ to B if and only if $\sum t_n$ converges and $||t_1, t_2, \ldots|| < +\infty$ (this is Theorem III in [4]). Moreover,

$$||T|| = ||t_1, t_2, \ldots||.$$

Assume now that the t_{mn} in (1) are bounded operators. Applying the Banach-Steinhaus theorem to the sequence of operators

$$T_m X = \sum_n t_{mn} x_n, \quad X = \{x_n\},$$

we obtain immediately Theorem 1 for our special case.

[Remark during the proof (December 19, 1952): essentially the same proof was given by K. Zeller in his paper [*Verallgemeinerte Matrixtransformationen*, Math. Z., vol. 56 (1952), 18–20] which appeared after this note was accepted for publication. However, Zeller leaves open the case of Robinson's general theorem.]

We note also the following theorem, whose proof follows along the same lines as [2, theorem 1].

THEOREM 2. *The transformation* (1) *has the property* $\sum ||y_m|| < +\infty$ *whenever* $\sum ||x_n|| < +\infty$ *if and only if, for some constant* M,

$$\sum_{m=1}^{\infty} ||t_{mn}\, x|| \leqslant M \quad \text{for } ||x|| \leqslant 1, \qquad n = 1, 2, \ldots . \tag{7}$$

3. Unbounded Operators

We now give the theorems referred to in the introduction.

THEOREM 3. *Let* B *be a Banach space and* L_m $(m = 1, 2, \ldots)$ *closed linear subspaces of* B. *Let* T_m *be a linear operator mapping* B *into itself, such that* T_m *is bounded on* L_m. *Let* $T_1 x, T_2 x, \ldots$ *be a bounded sequence for each* $x \in B$. *Then there is an* N *such that all* T_m *are bounded on* $\bigcap_{m=1}^{N} L_m$.

Proof. We may assume without loss of generality that $L_1 \supset L_2 \supset \ldots$, for otherwise we should consider

$$L_m^* = \bigcap_{\mu=1}^{m} L_\mu .$$

Suppose the conclusion does not hold. Then for each L_m there is a T_n which is not bounded on L_m, where necessarily $n > m$. Passing to a subsequence of the T_m, we may assume T_{m+1} is not bounded on L_m, for each m.

Let H_m be the bound of T_m on L_m. Choose a sequence $\alpha_1, \alpha_2, \ldots$ with

$$H_{m+1} \sum_{\mu > m} \alpha_\mu \leqslant 1. \tag{8}$$

Choose a sequence $x_1, x_2, \ldots$ of elements of B inductively, as follows. If $x_1, \ldots, x_{m-1}$ are already known, choose x_m such that $x_m \in L_m$, $||x_m|| \leqslant 1$, and

$$\alpha_m ||T_{m+1}\, x_m|| > m + \sum_{\mu=1}^{m-1} \alpha_\mu ||T_{m+1}\, x_\mu||. \tag{9}$$

We put

$$x = \sum_{\mu=1}^{\infty} \alpha_\mu\, x_\mu \in B;$$

then

$$T_{m+1}\, x = \alpha_n\, T_{m+1}\, x_m + \sum_{\mu=1}^{m-1} \alpha_\mu\, T_{m+1}\, x_\mu + T_{m+1} \sum_{\mu=m+1}^{\infty} \alpha_\mu\, x_\mu. \tag{10}$$

Here $x_\mu \in L_{m+1}$ for $\mu \geqslant m+1$, and so

$$\sum_{m+1}^{\infty} \alpha_\mu\, x_\mu \in L_{m+1}.$$

Hence

$$||T_{m+1}\sum_{\mu>m}\alpha_\mu x_\mu|| \leqslant H_{m+1}\sum_{\mu>m}\alpha_\mu \leqslant 1. \tag{11}$$

From the relations (10), (9), (11), (8) it follows that $\{T_m x\}$ is unbounded. This contradiction proves the theorem.

THEOREM 4. *Let $L_1 \subset L_2 \subset \ldots$ be closed linear subspaces of a Banach space B. Let $T_m = S_m + T_m'(m = 1, 2, \ldots)$, where T_m, S_m, T_m' are linear operators mapping B into itself, each T_m' is bounded, and S_m vanishes outside of L_m. Let $\{T_m x\}$ be bounded for each $x \in B$. Then there is an N such that all T_m are bounded on the part of the unit sphere not contained in L_N.*

The proof is a variation of that of Theorem 3. Let U be the unit sphere of B. We may assume that T_{m+1} is not bounded on $U - L_m$. We take $H_m = ||T_m'||$, and define the α_m as before. The x_m are defined inductively such that

$$x_m \in U - L_m, \quad \alpha_m||T_{m+1}\, x_m|| > m + \sum_{\mu=1}^{m-1}\alpha_\mu||T_{m+1}\, x_\mu||,$$

$$\alpha_m||x_m|| < \max\left\{\frac{1}{3}d_{m-1}, \frac{1}{9}d_{m-2}, \ldots, \frac{1}{3^{m-1}}d_1\right\},$$

where $d_\mu = \rho(\alpha_\mu x_\mu, L_\mu) > 0$ is the distance from $\alpha_\mu x_\mu$ to L_μ.

Then for

$$z_m = \sum_{\mu=m+1}^{\infty}\alpha_\mu x_\mu,$$

$$\rho(z_m, L_{m+1}) \geqslant d_{m+1} - \sum_{\mu=m+2}^{\infty}\alpha_\mu||x_\mu|| \geqslant d_{m+1} - \frac{1}{3}d_{m+1} - \frac{1}{9}d_{m+1} - \ldots$$

$$= \frac{1}{2}d_{m+1} > 0.$$

This shows that $z_m \notin L_m$. Therefore

$$||T_{m+1}\, z_m|| = ||T_{m+1}'\, z_m|| \leqslant H_{m+1}\sum_{m+1}^{\infty}\alpha_\mu \leqslant 1,$$

and the proof is completed by applying (10).

Theorems 3 and 4 remain true even if T_m maps B into a Banach space B_m depending on m. A similar remark applies to Theorems 1 and 2.

4. THE PROOF OF THEOREM 1

We begin with the following remark. Let $t_1, t_2, \ldots$ be a sequence of linear operators, mapping B into itself. If $\{t_n x_n\}$ is a bounded sequence for each $\{x_n\} = X \in c(B)$, then there is a number ν such that each

operator t_n with $n > \nu$ is bounded. For otherwise there would be indices $n_1 < n_2 < \ldots$ and elements x_{n_k} with

$$||x_{n_k}|| \leqslant 1 \text{ and } ||t_{n_k} x_{n_k}|| \to +\infty.$$

Then if

$$y_k = x_{n_k} ||t_{n_k} x_{n_k}||^{-\frac{1}{2}},$$

we should have

$$y_k \to 0 \text{ and } ||t_{n_k} y_k|| \to +\infty,$$

a contradiction. This proves our statement.

If now an operator of the form $T_m X = \sum_n t_{mn} x_n$ is defined for each $X \in c(B)$, it follows that there is a closed linear subspace L_m of $c(B)$, viz., the set of sequences with $x_1 = x_2 = \ldots = x_\nu = 0$, on which T_m is bounded. Hence, by Theorem 3, there is an N such that each T_m ($m = 1, 2, \ldots$) is bounded on the set of sequences with $x_1 = x_2 = \ldots = x_N = 0$. Clearly this implies that all t_{mn} are bounded for $n > N$. By combining this with the special case of Theorem 1 proved in §2, we obtain the complete proof of Theorem 1.

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SECTION THREE

The Extended Euler-Knopp Transformation

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Summation methods based on the following transformations

$$G_1(p): \qquad b_n = \sum_{k=n}^{\infty} \binom{k}{n} p^{k-n}(1-p)^n a_k,$$

$$G_2(p): \qquad b_n = \sum_{k=0}^{\infty} \binom{k+n-1}{n} p^k(1-p)^n a_k,$$

have recently been studied by W. Meyer-König [3] and P. Vermes [6]. Here p is a real or complex constant, and the series $\sum a_k$ is said to be $G_j(p)$-summable to s $(j = 1, 2)$ if the series $\sum b_n$ exists and converges to s. It was noted in [3] that these methods may be generated by the functions

$$g_1(t) = p + (1-p)t,$$

$$g_2(t) = p/\{1 - (1-p)t\},$$

respectively, by arranging $\sum_k a_k\{g_j(t)\}^k$ formally in powers of t and putting $t = 1$; this yields the series $\sum b_n$ in each case. If we take

$$g_3(t) = pt/\{1 - (1-p)t\}$$

we obtain the method

$$G_3(p): \quad b_0 = a_0, \; b_n = \sum_{k=1}^{n} \binom{n-1}{k-1} p^k(1-p)^{n-k} a_k \qquad (n = 1, 2, \ldots)$$

which can be shown by summing and differencing to be identical with the Euler-Knopp sequence-to-sequence method

$$E(p): \qquad \sigma_n = \sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k} s_k.$$

Thus $G_1(p)$, $G_2(p)$, $G_3(p)$ are special cases of the method G in which we take any suitable function

$$g(t) = \sum_{n=0}^{\infty} c_n t^n,$$

with $g(1) = 1$, let $\sum a_k\{g(t)\}^k = \sum b_n t^n$ formally, and call $\sum a_k$ G-summable to s if $\sum b_n = s$. If we put

$$\{g(t)\}^k = \sum_{n=0}^{\infty} c_n^{(k)} t^n$$

(so that $c_n = c_n^{(1)}$), we may define the series-to-series method G by the transform

$$G: \qquad b_n = \sum_{k=0}^{\infty} c_n^{(k)} a_k$$

If $c_0 = 0$ we have a row-finite matrix; otherwise the rows are infinite. The case $c_0 = 0$, $c_n > 0$ $(n = 1, 2, \ldots)$ was studied by O. Perron [4]; and the same with $c_n > 0$ relaxed, by the author [2]. K. Knopp [1] allowed $c_0 \neq 0$, and determined the w-region in which the series $\sum a_n w^n$ is G-summable, under certain conditions on $g(t)$, knowledge of the singularities of the function defined by $\sum a_n w^n$ being assumed (the analogue of the Borel polygon theorem).

The cases $c_0 = 0$ and $c_0 \neq 0$ require somewhat different treatment. The aim of the present note is to continue the study of the case $c_0 \neq 0$, giving the new form of some of the theorems in [2], and pointing out the application to the methods $G_1(p)$ and $G_2(p)$. Let us write out more fully the rearrangement involved in the method G:

$$\begin{array}{ll} & a_0 = a_0 \\ & a_1\, g(t) = a_1\, c_0^{(1)} + a_1\, c_1^{(1)} t + a_2\, c_2^{(1)} t^2 + \ldots \\ (A) \qquad & a_2\{g(t)\}^2 = a_2\, c_0^{(2)} + a_2\, c_1^{(2)} t + a_2\, c_2^{(2)} t^2 + \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \hline & \sum a_k\{g(t)\}^k = b_0 \quad + b_1 t \quad + b_2 t^2 \quad + \ldots \end{array}$$

We see at once that the assumption that b_n is defined for each n implies that each column forms a convergent series, and in particular, since $c_0^{(k)} = c_0^k$, the series $\sum a_k z^k$ has radius of convergence at least $|c_0|$. Furthermore, the assumption that $\sum b_n$ converges implies that $\sum b_n t^n$ has radius of convergence at least 1. We shall also assume that $g(1) = 1$, that the series $\sum c_n t^n$ has radius of convergence greater than 1, and that $z = g(t)$ maps $|t| = 1$ into a simple closed Jordan curve J in the z plane, and maps $|t| < 1$ simply on D, the interior of J.

THEOREM 1. *Let $\sum a_n z^n$ have radius of convergence $r > |c_0|$. Then if $\sum a_n$ is G-summable, the function $\psi(z) = \sum a_n z^n$ can be continued analytically through D, and conversely if $\psi(z)$ can be continued analytically through D', the closure of D, then $\sum a_n$ is G-summable.*

Remark. This can be sharpened somewhat (see [3, theorem 4] for $G_1(p)$) but the present form is sufficient for our purposes.

Proof. It follows from our hypothesis by the Weierstrass double-series theorem that each column of (A) converges, and if $\phi(t) = \psi\{g(t)\}$ the expansion $\phi(t) = b_0 + b_1 t + \ldots$ is valid at least for sufficiently small values of t. If $\sum a_n$ is G-summable, $\phi(t)$ is defined for $|t| < 1$; then, if $t = h(z)$, the [function $\phi\{h(z)\}$ gives the continuation of $\psi(z)$ through D. Conversely, if $\psi(z)$ can be continued analytically through D', then $\phi(t)$ is analytic on $|t| \leqslant 1$, and $\sum b_n t^n$ converges when $t = 1$. This completes the proof.

As an example, if $a_n = w^n$ the function $\psi(z) = \sum w^n z^n$ has radius of convergence $|1/w|$ and a pole at $z = 1/w$. Thus for $\sum w^n$ to be G-summable it is sufficient that $|1/w| > |c_0|$ and that $1/w$ does not lie in D', that is, $|h(1/w)| > 1$. Necessary conditions are $|1/w| > |c_0|$ and $|h(1/w)| \geqslant 1$; whether $|h(1/w)| > 1$ is necessary depends on G. For $G_1(p)$ and $G_2(p)$ see Vermes [6, pp. 550, 558]. For the summability of $\sum a_n w^n$ in the case where J contains the origin and is cut by each half-ray from the origin in one point only, see Knopp [1].

The method G is regular if all $c_n \geqslant 0$; the proof given by Perron [4, theorem 1] with $c_0 = 0$ is valid for $c_0 \neq 0$. If not all $c_n \geqslant 0$, the method G need not be regular; it may however be regular over certain classes of series as shown by Theorems 2, 2a, 3 that follow.

THEOREM 2. *If $R > |c_0|$, a necessary and sufficient condition for G to have the property that $\sum a_n$ is G-summable whenever $\sum a_n z^n$ has its radius of convergence greater than R, is that D' should lie in the circle $|z| \leqslant R$.*

This follows from Theorem 1. (We remark that if $R < 1$ the theorem is vacuous since J passes through $z = 1$.) More generally, if K is any closed domain containing the circle $|z| \leqslant c_0$ in its interior, we have

THEOREM 2a. *A necessary and sufficient condition for G to have the property that $\sum a_n$ is G-summable whenever $\sum a_n z^n$ represents a function analytic in K, is that D' should lie in K.*

This contains the main part of a theorem given by J. Teghem [5] for Euler-Knopp summability.

THEOREM 3. *Assume $|c_0| < 1$. Then G is consistent with convergence; that is, if $\sum a_n$ and $\sum b_n$ both converge, they converge to the same value.*

Proof. If $\sum a_n z^n$ converges for $z = 1$, its radius of convergence is greater than $|c_0|$, so the considerations of Theorem 1 apply. The result follows by Abel's theorem on power series. Similarly in Theorem 2, $\sum a_n$ is G-summable to its ordinary value.

THEOREM 4. *Let $\sum a_n$ be G-summable, and let $\sum a_n z_n$ have radius of convergence greater than $|c_0|$. Then $\sum a_n w^n$ is G-summable for all values of w such that $wD' \subset D$.*

Proof. We write $\Psi(z) = \sum a_n w^n z^n$. By Theorem 1, $\sum a_n w^n$ is G-summable if $\sum a_n w^n z^n$ has radius of convergence greater than $|c_0|$ and $\Psi(z)$ is analytic in D'. Now $wD' \subset D$ implies $|w| < 1$, so the radius of convergence of $\sum a_n w^n z^n$ is greater than that of $\sum a_n z^n$, hence greater than $|c_0|$. Also $\Psi(z) = \psi(wz)$, and since ψ is known to be analytic in D, the result follows.

THEOREM 5. *Let the series $\sum a_n$, $\sum a_n'$ be G-summable to s, s', and let $\sum a_n z^n$, $\sum a_n' z^n$, have radii of convergence greater than $|c_0|$. Let $\sum a_n''$ be the Cauchy product, $a_n'' = a_0 a_n' + \ldots + a_n a_0'$. Then $\sum a_n''$ is G-summable to ss', provided $\sum b_n$, the G-transform of $\sum a_n$, is absolutely convergent.*

For if $\sum b_n'$, $\sum b_n''$ are the G-transforms of $\sum a_n'$, $\sum a_n''$, then $\sum b_n''$ is none other than the Cauchy product of $\sum b_n$, $\sum b_n'$. The result follows by Merten's theorem.

In conclusion we state briefly the results for $G_1(p)$ and $G_2(p)$. For $G_1(p)$ we have $c_0 = p$, and the region D is $|z - p| < |1 - p|$. If $\Re p < \frac{1}{2}$, D contains the origin and the shortest distance from J to the origin is $|1 - p| - |p|$. If $|p| < \frac{1}{2}|1 - p|$ we have $|1 - p| - |p| > |p|$, so the assumption that $\sum a_n z^n$ is analytic in D' implies that the radius of convergance of $\sum a_n z^n$ is greater than $|c_0|$; therefore $\sum a_n$ is $G_1(p)$-summable by Theorem 1. The condition in Theorem 2, that D' lies in $|z| \leqslant R$, becomes

$$|1 - p| + |p| \leqslant R.$$

The condition in Theorem 4, that $wD' \subset D$, becomes

$$|p/(1 - p)|\,|w - 1| + |w| < 1.$$

For $G_2(p)$ we require $|1 - p| < 1$. We have again $c_0 = p$, and the region D is $|z - p| < |z|\ |1 - p|$. This is a circle with centre $p/\{1 - |1 - p|^2\}$ and radius $|1 - p|\,|p|/\{1 - |1 - p|^2\}$. The condition in Theorem 2, that D' lies in $|z| \leqslant R$, becomes

$$|p/R| + |1 - p| \leqslant 1,$$

and the condition in Theorem 4, that $wD' \subset D$, becomes

$$|1/(1 - p)|\,|w - 1| + |w| < 1.$$

The corresponding results for $G_3(p)$ are given in [2].

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SECTION THREE

Representations of Positive Real Numbers by Infinite Sequences of Integers

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1. Introduction

OUR standard decimal notation is based on the following theorems of arithmetic:

1. Any positive integer N can be written uniquely in the form

$$N = a_0 + 10a_1 + 10^2a_2 + \ldots + 10^ra_r$$

where a_i are integers such that $0 \leqslant a_i \leqslant 9$, $i = 1, 2, 3, \ldots, r$. In other words, the integer N determines the number r and each of the a_i.

2. Any real number M, $0 < M < 1$, can be written uniquely in the form

$$M = \frac{c_1}{10} + \frac{c_2}{10^2} + \ldots + \frac{c_r}{10^r} + \ldots$$

where c_i are integers such that $0 \leqslant c_i \leqslant 9$ for $i = 1, 2, 3, \ldots$, and furthermore, for infinitely many i, $c_i \neq 9$. If we remove the latter part of the condition, uniqueness is destroyed, for example, the numbers 0.5000 . . . and 0.49999 . . . are identical.

These theorems are readily generalized to the case where the number 10 is replaced by the number a, where a is any integer $\geqslant 2$. Less trivial generalizations have been given in [2] by Faber who has shown that:

1. Every integer can be expressed uniquely in the form

$$b_1 + b_2\,2! + b_3\,3! + \ldots + b_n\,n!$$

where b_i are integers such that $0 \leqslant b_i \leqslant i$.

2. Every number M, $0 < M < 1$, can be expressed uniquely in the form

$$M = \frac{c_1}{2!} + \frac{c_2}{3!} + \frac{c_3}{4!} + \ldots + \frac{c_r}{(r+1)!} + \ldots$$

where c_i are integers such that $0 \leqslant c_i \leqslant i$ and for infinitely many i, $c_i < i$. Faber obtains, from this representation, a fascinating proof of the countability of the rational numbers. In [1] Cantor represents positive numbers less than one by expressions of the form

$$\frac{\lambda}{b} + \frac{\mu}{bb'} + \frac{\nu}{bb'b''} + \dots$$

where $\lambda, \mu, \nu, \dots$ are integers such that $0 \leqslant \lambda \leqslant b - 1, 0 \leqslant \mu \leqslant b' - 1$, etc. In this paper, we will show that, in a sense, Cantor's representation is the most general possible. Our purpose is actually to find all possible representations of real numbers by sequences of integers, of the type described in this introduction.

2. The Main Theorem

Let $\alpha_0 = 1, \alpha_1, \alpha_2, \dots, \alpha_n, \dots$ be a sequence of real numbers strictly increasing to infinity; that is $\alpha_{i+1} > \alpha_i$ for $i = 0, 1, 2, \dots$ and for any real number N, there is an integer n such that $\alpha_n > N$. Let $\beta_1, \beta_2, \dots, \beta_n, \dots$ be a second sequence of real numbers strictly increasing to infinity. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be any sequence of positive integers and let $\mu_1, \mu_2, \dots, \mu_n$ be any second sequence of positive integers, including the possibility that $\mu_i = \lambda_i$. We ask, what are the necessary and sufficient conditions in order that every positive real number N may be expressed uniquely in the form

$$N = a_n \alpha_n + a_{n-1} \alpha_{n-1} + \dots + a_0 \alpha_0 + \frac{b_1}{\beta_1} + \dots + \frac{b_i}{\beta_i} + \dots$$

where $a_0, a_1, \dots, a_n, b_1, b_2, \dots$ are integers such that

$$0 \leqslant a_i \leqslant \lambda_i, \quad 0 \leqslant b_i \leqslant \mu_i$$

and such that for infinitely many i, $b_i < \mu_i$.

In order to simplify our notation we will replace μ_n by λ_{-n} and $1/\beta_n$ by α_{-n} when these are more convenient. Then $\alpha_r \to 0$ as $r \to -\infty$ and $\alpha_r \to \infty$ as $r \to \infty$. We will also use a reverse summation notation, i.e., we will use

$$*\sum_{i=r}^{s} u_i$$

for the sum

$$u_r + u_{r-1} + \dots + u_s.$$

This notation will only be used in the case where $r \geqslant s$ and the order of summation will only be important when $s = -\infty$.

THEOREM 1. *A necessary and sufficient condition that every real positive number N be expressible uniquely in the form*

$$N = {}^{*}\sum_{i=r}^{-\infty} a_i \alpha_i$$

where $0 \leqslant a_i \leqslant \lambda_i$ and for infinitely many i, $a_i < \lambda_i$ is that

$$\alpha_{k+1} = {}^{*}\sum_{i=k}^{-\infty} \lambda_i \alpha_i \qquad \text{for } -\infty < k < \infty.$$

Proof of sufficiency. Assume

$$\alpha_{k+1} = {}^{*}\sum_{i=k}^{-\infty} \lambda_i \alpha_i \qquad \text{for } -\infty < k < \infty.$$

Hence

$$\alpha_0 = {}^{*}\sum_{i=-1}^{-\infty} \lambda_i \alpha_i = 1.$$

For $n > 0$,

$$\alpha_n = {}^{*}\sum_{i=n-1}^{0} \lambda_i \alpha_i + {}^{*}\sum_{i=-1}^{-\infty} \lambda_i \alpha_i = {}^{*}\sum_{i=n-1}^{0} \lambda_i \alpha_i + 1.$$

We first show that α_i is an integer for all $i \geqslant 0$. We have $\alpha_0 = 1$, an integer, and assuming $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are integers, the equation

$$\alpha_n = {}^{*}\sum_{i=n-1}^{0} \lambda_i \alpha_i + 1$$

shows that α_n is also an integer since all the λ_i are integral. Now, let N be any positive real number. Put $N = K + L$ where K is an integer and $0 \leqslant L < 1$. We will represent K in the form

$$K = {}^{*}\sum_{i=r}^{0} a_i \alpha_i$$

where a_i are integers such that $0 \leqslant a_i \leqslant \lambda_i$, and L in the form

$$L = {}^{*}\sum_{i=-1}^{-\infty} a_i \alpha_i$$

where again $0 \leqslant a_i \leqslant \lambda_i$, but for infinitely many i, $a_i < \lambda_i$. To show that an integer K is representable as described above we will use complete induction. $1 = \alpha_0$, hence 1 is representable. Assume that all integers $T < \alpha_i$ are representable in the form

$$T = {}^{*}\sum_{r=i-1}^{0} b_r \alpha_r.$$

Let $\alpha_i \leqslant K < \alpha_{i+1}$. Consider the intervals

$$\alpha_i \leqslant x < 2\alpha_i,\ 2\alpha_i \leqslant x < 3\alpha_i,\ 3\alpha_i \leqslant x < 4\alpha_i, \ldots,$$
$$(\lambda_i - 1)\alpha_i \leqslant x < \lambda_i \alpha_i,\ \lambda_i \alpha_i \leqslant x < \alpha_{i+1}.$$

Since these intervals cover the range $\alpha_t \leqslant x < \alpha_{t+1}$, K must lie in one of these intervals. If $r\alpha_t \leqslant K < (r+1)\alpha_t$, $1 \leqslant r < \lambda_t$ then $0 \leqslant K - r\alpha_t < \alpha_t$. By hypothesis

$$K - r\alpha_t = {}^*\sum_{t=t-1}^{0} c_t \alpha_t.$$

Hence

$$K = {}^*\sum_{t=t}^{0} c_t \alpha_t$$

where $1 \leqslant c_t = r \leqslant \lambda_t - 1$. If K lies in the last interval, i.e. $\lambda_t \alpha_t \leqslant K < \alpha_{t+1}$, then

$$0 \leqslant K - \lambda_t \alpha_t < \alpha_{t+1} - \lambda_t \alpha_t$$

$$= 1 + {}^*\sum_{t=t}^{0} \lambda_t \alpha_t - \lambda_t \alpha_t = 1 + {}^*\sum_{t=t-1}^{0} \lambda_t \alpha_t = \alpha_t.$$

Since $K - \lambda_t \alpha_t < \alpha_t$, by our induction hypothesis

$$K - \lambda_t \alpha_t = {}^*\sum_{t=t-1}^{0} c_t \alpha_t.$$

Hence

$$K = {}^*\sum_{t=t}^{0} c_t \alpha_t$$

where $c_t = \lambda_t$. We now come to the representation of L where $0 \leqslant L < 1$. We define a sequence $a_{-1}, a_{-2}, a_{-3}, \ldots, a_{-n}, \ldots$ as follows. First, determine the positive (or zero) integer a_{-1} by the condition that a_{-1} is the maximal admissible integer such that $a_{-1}\alpha_{-1} \leqslant L$. The maximal admissible integer a_{-1} is defined as follows. If $\lambda_{-1}\alpha_{-1} \leqslant L$ take $a_{-1} = \lambda_{-1}$. If $\lambda_{-1}\alpha_{-1} > L$ we choose a_{-1} to be the unique positive integer satisfying both the inequalities $a_{-1}\alpha_{-1} \leqslant L$ and $(a_{-1}+1)\alpha_{-1} > L$. In any case, $0 \leqslant a_{-1} \leqslant \lambda_{-1}$. We then choose a_{-2} as the maximal admissible integer satisfying $a_{-1}\alpha_{-1} + a_{-2}\alpha_{-2} \leqslant L$. In general, a_{-t} is the maximal admissible integer satisfying

$$a_{-1}\alpha_{-1} + a_{-2}\alpha_{-2} + \ldots + a_{-t}\alpha_{-t} \leqslant L.$$

Since for all i,

$${}^*\sum_{t=-1}^{i} a_t \alpha_t \leqslant L$$

the series

$${}^*\sum_{t=-1}^{-\infty} a_t \alpha_t$$

is convergent and

$$L \geqslant {}^{*}\sum_{i=-1}^{-\infty} a_i \alpha_i.$$

We next show that for infinitely many i, $a_i < \lambda_i$. In the first place, not all $a_i = \lambda_i$ since

$${}^{*}\sum_{i=-1}^{-\infty} \lambda_i \alpha_i = \alpha_0 = 1 > L.$$

Let r be the first index such that $a_{-r} < \lambda_{-r}$. By the maximality condition on a_{-r} we have

$${}^{*}\sum_{i=-1}^{-r} a_i \alpha_i \leqslant L < {}^{*}\sum_{i=-1}^{-r} a_i \alpha_i + \alpha_{-r}.$$

We now show that there is an $r_1 > r$ such that

$$a_{-r_1} < \lambda_{-r_1}.$$

If this were not so, then for all $i > r$, $a_{-i} = \lambda_{-i}$, from which it follows that

$$L \geqslant {}^{*}\sum_{i=-1}^{-\infty} a_i \alpha_i = {}^{*}\sum_{i=-1}^{-r} a_i \alpha_i + {}^{*}\sum_{i=-r-1}^{-\infty} \lambda_i \alpha_i = {}^{*}\sum_{i=-1}^{-r} a_i \alpha_i + \alpha_{-r} > L,$$

a contradiction. By continuation of this process one can obtain an infinite sequence

$$r < r_1 < r_2 < r_3 \ldots < r_n < \ldots$$

having the property that

$$a_{-r_i} \neq \lambda_{-r_i}$$

Furthermore, for any such r_i,

$${}^{*}\sum_{i=-1}^{-r_i} a_i \alpha_i \leqslant L < {}^{*}\sum_{i=-1}^{-r_i} a_i \alpha_i + \alpha_{-r_i}.$$

Since $\alpha_{-r_i} \to 0$ as $i \to \infty$ and since

$${}^{*}\sum_{i=-1}^{-\infty} a_i \alpha_i$$

is a series of positive terms, we have

$$L = {}^{*}\sum_{i=-1}^{-\infty} a_i \alpha_i.$$

We now show that the representation we have obtained for the number N is unique. If not, suppose there are two representations of N, namely

$$N = {}^{*}\sum_{j=r}^{-\infty} a_j \alpha_j = {}^{*}\sum_{j=s}^{-\infty} b_j \alpha_j$$

where $a_r \neq 0$, $b_s \neq 0$, $0 \leqslant a_j \leqslant \lambda_j$, $0 \leqslant b_j \leqslant \lambda_j$ and for infinitely many i, $a_i < \lambda_i$, for infinitely many k, $b_k < \lambda_k$. We will show that $r = s$ and $a_k = b_k$ for $r \geqslant k > -\infty$. If $r \neq s$, suppose $r > s$; then

$$*\sum_{j=r}^{-\infty} a_j \alpha_j \geqslant \alpha_r = *\sum_{j=r-1}^{-\infty} \lambda_j \alpha_j > *\sum_{j=s}^{-\infty} b_j \alpha_j,$$

a contradiction. Hence $r = s$. If now $a_r > b_r \geqslant 1$, then

$$*\sum_{j=r}^{-\infty} a_j \alpha_j \geqslant a_r \alpha_r = (a_r - 1)\alpha_r + \alpha_r$$

$$= (a_r - 1)\alpha_r + *\sum_{i=r-1}^{-\infty} \lambda_i \alpha_i > *\sum_{i=r}^{-\infty} b_i \alpha_i,$$

a contradiction. Hence $a_r = b_r$ and

$$*\sum_{j=r-1}^{-\infty} a_j \alpha_j = *\sum_{j=r-1}^{-\infty} b_j \alpha_j.$$

As before, $a_{r-1} = b_{r-1}$ and, in general, by inudction $a_i = b_i$ for $r \geqslant i > -\infty$.

Proof of necessity. Assume that the condition

$$\alpha_{r+1} = *\sum_{i=r}^{-\infty} \lambda_i \alpha_i, \qquad -\infty < r < \infty$$

is false. The following possibilities arise:

Case 1. There is at least one r for which

$$\alpha_{r+1} > *\sum_{i=r}^{-\infty} \lambda_i \alpha_i.$$

Case 2. For all r, it is false that

$$\alpha_{r+1} > *\sum_{i=r}^{-\infty} \lambda_i \alpha_i.$$

Case (2) breaks up into 2 sub-cases namely:

Case $(2)_1$.

$$*\sum_{i=r}^{-\infty} \lambda_i \alpha_i$$

is divergent for $-\infty < r < \infty$ and,

Case $(2)_2$.

$$\alpha_{r+1} \leqslant *\sum_{i=r}^{-\infty} \lambda_i \alpha_i$$

for $-\infty < r < \infty$ and for at least one k,

$$\alpha_{k+1} < {}^*\sum_{i=k}^{-\infty} \lambda_i \alpha_i.$$

In case (1) let T be a number such that

$$\alpha_{r+1} > T > {}^*\sum_{i=r}^{-\infty} \lambda_i \alpha_i.$$

By definition, T has no representation. In case $(2)_1$, i.e. when

$${}^*\sum_{i=r}^{-\infty} \lambda_i \alpha_i$$

is divergent, define numbers $a_r, a_{r-1}, a_{r-2}, \ldots$ by saying that a_r is the maximal admissible integer such that $a_r \alpha_r \leqslant \alpha_{r+1}$; a_{r-1} is the maximal admissible integer such that $a_r \alpha_r + a_{r-1} \alpha_{r-1} \leqslant \alpha_{r+1}$ etc. Hence, by definition

$${}^*\sum_{i=r}^{-\infty} a_i \alpha_i$$

is convergent and

$$\alpha_{i+1} \geqslant {}^*\sum_{i=r}^{-\infty} a_i \alpha_i.$$

Since

$${}^*\sum_{i=s}^{-\infty} \lambda_i \alpha_i$$

is divergent, infinitely many $a_i < \lambda_i$. Let $i_1 < i_2 < i_3 < i_4 \ldots$ be an infinite sequence of integers such that

$$a_{-i_k} < \lambda_{-i_k}.$$

As before, we have

$${}^*\sum_{i=r}^{-i_k} a_i \alpha_i \leqslant \alpha_{r+1} < {}^*\sum_{i=r}^{-i_k} a_i \alpha_i + \alpha_{-i_k}.$$

Hence

$$0 \leqslant \alpha_{r+1} - {}^*\sum_{i=r}^{-i_k} a_i \alpha_i < \alpha_{-i_k}.$$

Since $\alpha_{-i_k} \to 0$ as $i_k \to \infty$, we have

$$\alpha_{r+1} = {}^*\sum_{i=r}^{-\infty} a_i \alpha_i.$$

We thus have two representations for α_{r+1}, namely

$$\alpha_{r+1} = {}^*\sum_{i=r}^{-\infty} a_i \alpha_i$$

and

$$\alpha_{r+1} = 1\alpha_{r+1} + 0\alpha_r + 0\alpha_{r-1} + \ldots.$$

In the case $(2)_2$ we are given that

$$\alpha_{k+1} < {}^*\sum_{i=k}^{-\infty} \lambda_i \alpha_i, \quad \alpha_{r+1} \leqslant {}^*\sum_{i=r}^{-\infty} \lambda_i \alpha_i, \qquad -\infty < r < \infty.$$

As before, define $a_k, a_{k-1}, a_{k-2}, \ldots$ by saying that a_k is the maximal admissible integer such that $a_k \alpha_k \leqslant \alpha_{k+1}$; a_{k-1} is the maximal admissible integer such that $a_k \alpha_k + a_{k-1} \alpha_{k-1} \leqslant \alpha_{k+1}$ etc. Then

$$\alpha_{k+1} \geqslant {}^*\sum_{i=k}^{-\infty} a_i \alpha_i.$$

At least one $a_i \neq \lambda_i$ and hence, $a_i < \lambda_i$, since

$$\alpha_{k+1} < {}^*\sum_{i=k}^{-\infty} \lambda_i \alpha_i.$$

If there were only a finite number of i for which $a_i < \lambda_i$ let $n_0 + 1$ be the last index for which $a_i < \lambda_i$. That is, we assume that

$$a_{n_0+1} < \lambda_{n_0+1}$$

but that $a_i = \lambda_i$ for all $i \leqslant n_0$ (remembering that in the series ${}^*\sum a_i \alpha_i$ the i run in decreasing order of magnitude). Then

$$\alpha_{k+1} \geqslant {}^*\sum_{i=k}^{n_0+1} a_i \alpha_i + {}^*\sum_{i=n_0}^{-\infty} \lambda_i \alpha_i \geqslant {}^*\sum_{i=k}^{n_0+1} a_i \alpha_i + \alpha_{n_0+1}.$$

But since a_{n_0+1} is a maximal admissible number and

$$a_{n_0+1} < \lambda_{n_0+1},$$

we have

$${}^*\sum_{i=k}^{n_0+1} a_i \alpha_i \leqslant \alpha_{k+1} < {}^*\sum_{i=k}^{n_0+1} a_i \alpha_i + \alpha_{n_0+1},$$

a contradiction. Therefore, infinitely many $a_i < \lambda_i$. As before, α_{k+1} has at least two representations. We have shown that when the condition

$$\alpha_{r+1} = {}^*\sum_{i=r}^{-\infty} \lambda_i \alpha_i, \qquad -\infty < r < \infty,$$

is violated, either there exist positive real numbers with no representations or there exist positive real numbers with at least two representations. Hence, we cannot have uniqueness of representation for every positive real number if we violate the fundamental condition: i.e. the condition is necessary.

3. Some Subsidiary Theorems

Theorem 2. *The positive integers λ_i, $-\infty < i < \infty$, may be chosen arbitrarily and these uniquely determine the α_i and the β_i.*

Proof. It can easily be shown by induction that

$$\alpha_0 = 1, \quad \alpha_n = (1 + \lambda_0)(1 + \lambda_1) \ldots (1 + \lambda_{n-1})$$

and that

$$\beta_n = (1 + \lambda_{-1})(1 + \lambda_{-2}) \ldots (1 + \lambda_{-n})$$

THEOREM 3. *The numbers α_n and β_n are integers such that for $i \leqslant j$, $\alpha_i \mid \alpha_j$ and $\beta_i \mid \beta_j$.* ($x \mid y$ means x is a divisor of y.)

Proof. The result follows immediately from the formulae of Theorem 2.

THEOREM 4. *The numbers α_i and β_i may be taken as arbitrary sequences of integers subject only to the conditions $\alpha_0 = 1$, $\alpha_i \mid \alpha_{i+1}$, and $\beta_i \mid \beta_{i+1}$ for all i.*

Proof. If we choose

$$\lambda_n = \frac{\alpha_{n+1}}{\alpha_n} - 1,$$

and

$$\lambda_{-n} = \frac{\beta_n}{\beta_{n-1}} - 1, \qquad n \geqslant 2, \quad \lambda_{-1} = \beta_1 - 1.$$

it is easily verified that the fundamental condition is satisfied.

The result of the Theorem 4 shows that Cantor in [1] actually obtained the most general representation (for the fractional part of the number at least).

THEOREM 5. *A necessary and sufficient condition that every positive rational number have a finite expansion, is that for every integer N, there exists an n such that $N \mid \beta_n$.*

We note here that a proof of half of this theorem was obtained by Cantor in [1].

Proof of necessity. Suppose there is an integer N such that N does not divide β_n for $n = 1, 2, 3, \ldots$. We will show that $1/N$ cannot have a finite expansion. For suppose that

$$\frac{1}{N} = \sum_{i=1}^{k} \frac{a_i}{\beta_i} = \frac{u}{\beta_k},$$

since $\beta_1 \mid \beta_2 \mid \beta_3 \ldots \mid \beta_k$. Then $\beta_k = Nu$, that is, $N \mid \beta_k$ which is a contradiction.

Proof of sufficiency. Suppose that for each N, there is an n such that $N \mid \beta_n$. We need only consider the representation of rational numbers a/b such that $0 < a/b < 1$. Suppose $b \mid \beta_1$. Then $a/b = r/\beta_1$. If $r \leqslant \lambda_{-1}$, r/β_1 is the required representation. If $r > \lambda_{-1}$ then

$$\frac{a}{b} = \frac{r}{\beta_1} \geqslant \frac{\lambda_{-1} + 1}{\beta_1} = \alpha_{-1} + \alpha_{-1}\lambda_{-1}$$

$$= \alpha_{-1}\lambda_{-1} + {}^{*}\sum_{i=-2}^{-\infty} \lambda_i \alpha_i = {}^{*}\sum_{i=-1}^{-\infty} \lambda_i \alpha_i = \alpha_0 = 1,$$

a contradiction. Hence every number a/b such that $0 < a/b < 1$ and such that $b \mid \beta_1$ has a representation by one term. Suppose now that any fraction whose denominator divides β_r has a finite representation stopping at the term in β_r and let c/d be a fraction such that $0 < c/d < 1$ and such that $d \mid \beta_{r+1}$. Then $c/d = k/\beta_{r+1}$. If $k \leqslant \lambda_{-r-1}$, k/β_{r+1} is the required representation of c/d by one term. If $k > \lambda_{-r-1}$, divide k by $\lambda_{-r-1} + 1$ obtaining $k = q\ (\lambda_{-r-1} + 1) + s$ where $0 \leqslant s \leqslant \lambda_{-r-1}$. Then

$$\frac{c}{d} = \frac{k}{\beta_{r+1}} = \frac{q(\lambda_{-r-1} + 1) + s}{\beta_{r+1}} = \frac{q}{\beta_r} + \frac{s}{\beta_{r+1}}.$$

By hypothesis

$$\frac{q}{\beta_r} = \frac{c_1}{\beta_1} + \frac{c_2}{\beta_2} + \ldots + \frac{c_r}{\beta_r}$$

where $0 \leqslant c_i \leqslant \lambda_{-i}$, and hence,

$$\frac{c}{d} = \frac{c_1}{\beta_1} + \frac{c_2}{\beta_2} + \ldots + \frac{c_r}{\beta_r} + \frac{s}{\beta_{r+1}}.$$

THEOREM 6. *If the system of representation defined by the sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *represents all rational numbers by finite expansions, then* $\{\lambda_{-n}\}$ *is an unbounded sequence.*

Proof. If $\lambda_{-n} \leqslant M$ for all n, take p a prime number such that $p > M + 1$. For all n,

$$\beta_n = (1 + \lambda_{-1})(1 + \lambda_{-2}) \ldots (1 + \lambda_{-n}).$$

Since all the numbers $1 + \lambda_{-1}$, $1 + \lambda_{-2}$, . . . , $1 + \lambda_{-n}$ are less than or equal to $1 + M$, all the prime factors of β_n are less than or equal to $1 + M$. Hence p is not a divisor of any β_n. By Theorem 5, not all rational numbers have a finite expansion, a contradiction. Hence, $\{\lambda_{-n}\}$ is unbounded.

THEOREM 7. *Suppose that* $\beta_{n+1} \geqslant \beta_n^{\,n}$ *for all* $n \geqslant n_0$. *If a number* T *be represented by a bounded representation, that is,*

$$T = {}^{*}\sum_{i=r}^{-\infty} a_i \alpha_i,$$

and $a_i \leqslant A$ *for all* i *but infinitely many* $a_i \neq 0$, *then* T *is a transcendental number.*

Proof. Put

$$T_N = {}^*\sum_{i=r}^{-N} a_i \alpha_i.$$

Since infinitely many $a_i \neq 0$, $T_N < T$ for all N. Also,

$$T_N = {}^*\sum_{i=r}^{-N} a_i \alpha_i = \frac{K}{\beta_N}.$$

Then,

$$0 < T - \frac{K}{\beta_N} = T - T_N = {}^*\sum_{i=-N-1}^{-\infty} a_i \alpha_i = a_{-N-1}\alpha_{-N-1} + {}^*\sum_{i=-N-2}^{-\infty} a_i \alpha_i$$

$$< a_{-N-1}\alpha_{-N-1} + {}^*\sum_{i=-N-2}^{-\infty} \lambda_i \alpha_i = a_{-N-1}\alpha_{-N-1} + \alpha_{-N-1}$$

$$= \alpha_{-N-1}(a_{-N-1} + 1) < \frac{A+1}{\beta_{N+1}} < \frac{A+1}{\beta_N{}^N} \quad \text{for all } N \geqslant n_0.$$

By Liouville's theorem, the inequalities

$$0 < T - \frac{K}{\beta_N} < \frac{A+1}{\beta_N{}^N},$$

for all $N \geqslant n_0$, implies that T is transcendental.

COROLLARY. *Let $\beta_{n+1} \geqslant \beta_n{}^n$ for all $n \geqslant n_0$. If a number T be represented by*

$$T = {}^*\sum_{i=n}^{-\infty} a_i \alpha_i$$

and $a_i \geqslant \lambda_i - A$, all i, then T is transcendental. For if $T = K + L$ where K is integral and $0 \leqslant L \leqslant 1$, then, by Theorem 7, $1 - L$ is transcendental and hence so is T.

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